Infinite Games: Simple Strategies for Simple Specifications

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June 21, 2011

Infinite games

- Two players alternately play letters from Σ_1 and Σ_2 ad infinitum
- Result: ω-word, α over the alphabet $\Sigma = \Sigma_1 \times \Sigma_2$

Infinite games

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Player 1	
Player 2	

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Infinite games

Round	1	2
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Player 2	x_1	
	$\underbrace{}$	

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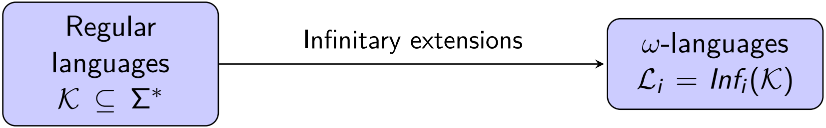
Game: $L \in \Sigma^\omega$, strategies: $K_1, K_2 \in \Sigma^*$

Languages and games

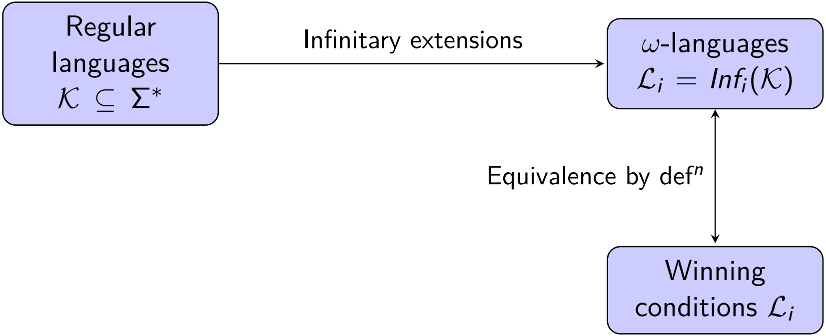
Languages and games

Regular
languages
$\mathcal{K} \subseteq \Sigma^*$

Languages and games



Languages and games



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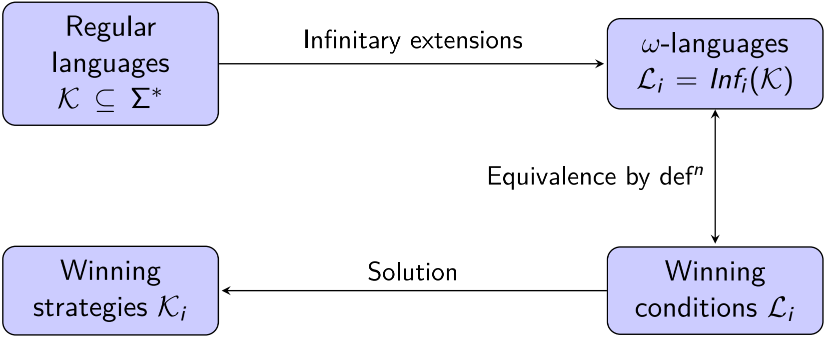

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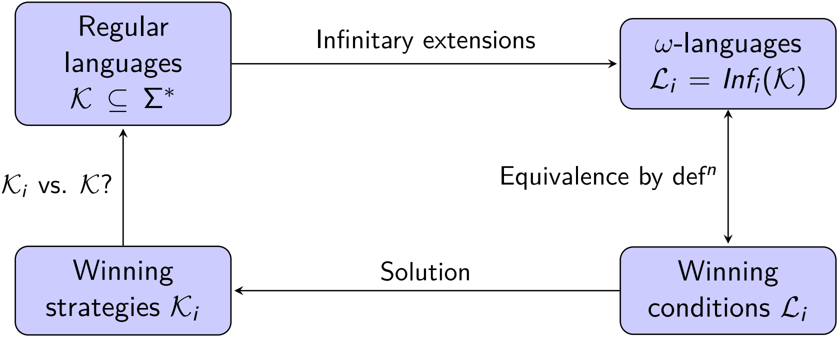

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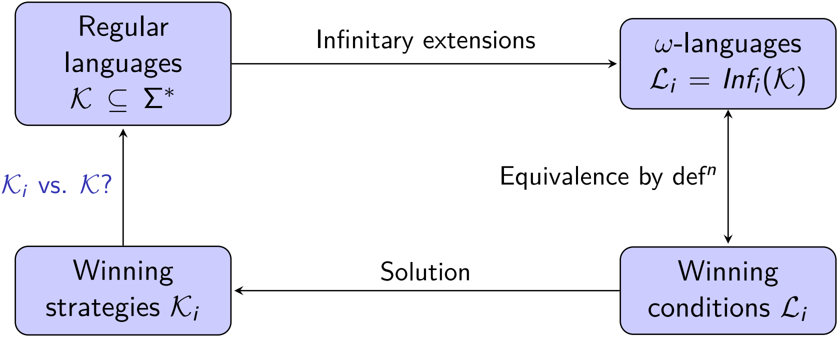

A simple game

$L =$ Only finitely many $\binom{a}{0} \wedge ($Infinitely many $\binom{b}{0} \Leftrightarrow$ Infinitely many $\binom{b}{1})$

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Player 2's winning strategy:

- Respond to a's with 0 only finitely often (or never)
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Motivation and results

Do simple games have simple strategies?

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Description languages

Winning strategies

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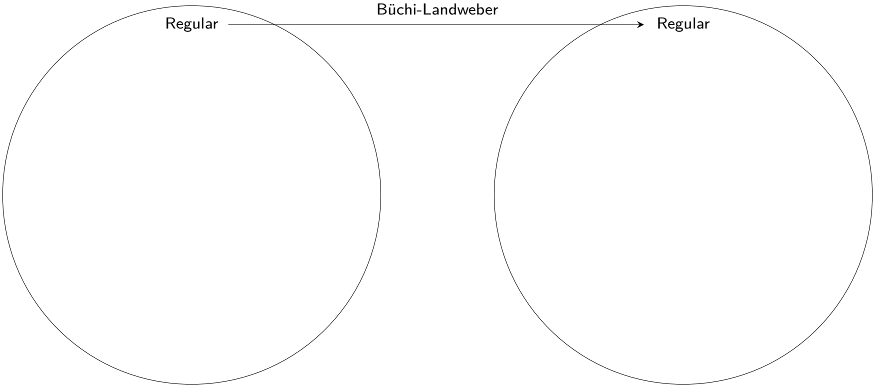

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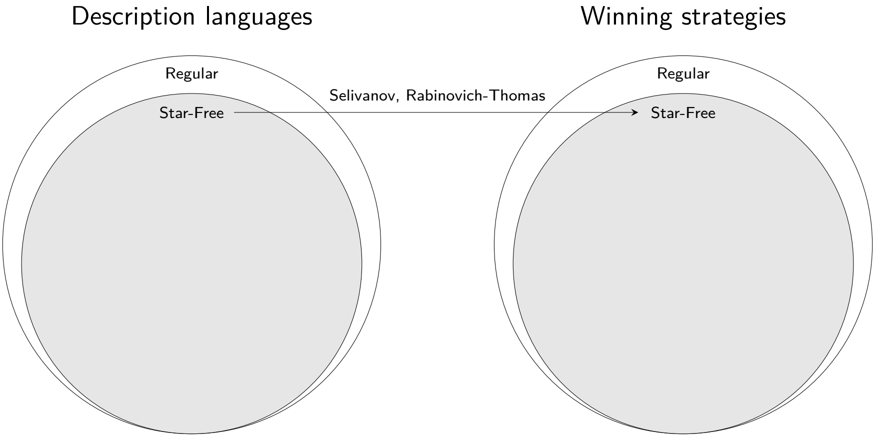

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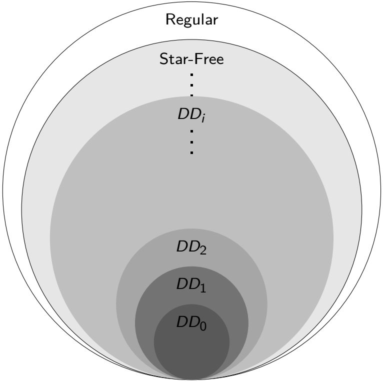
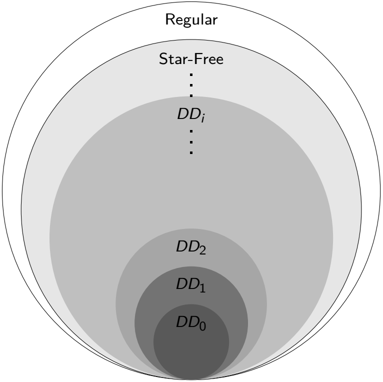


Winning strategies



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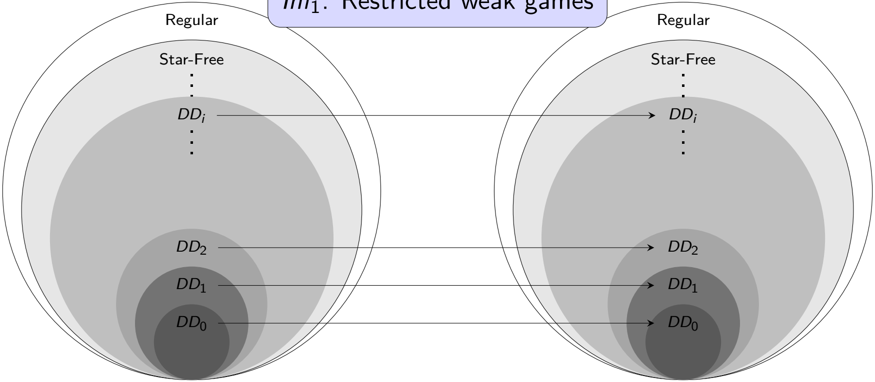
Inf_1: Restricted weak games



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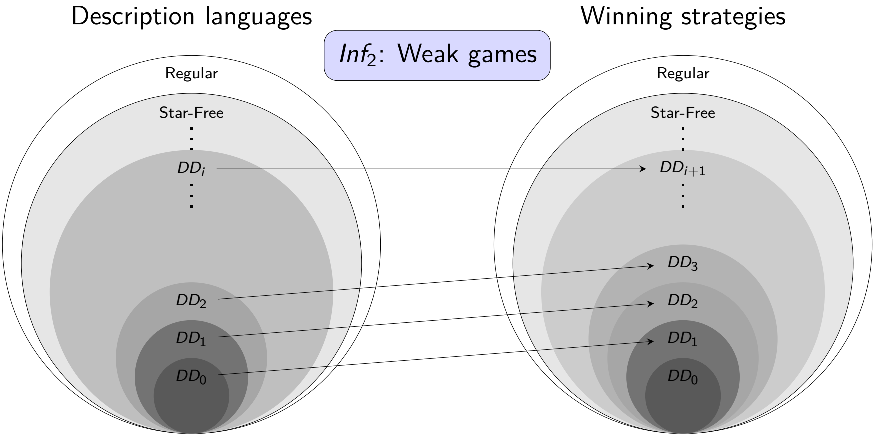

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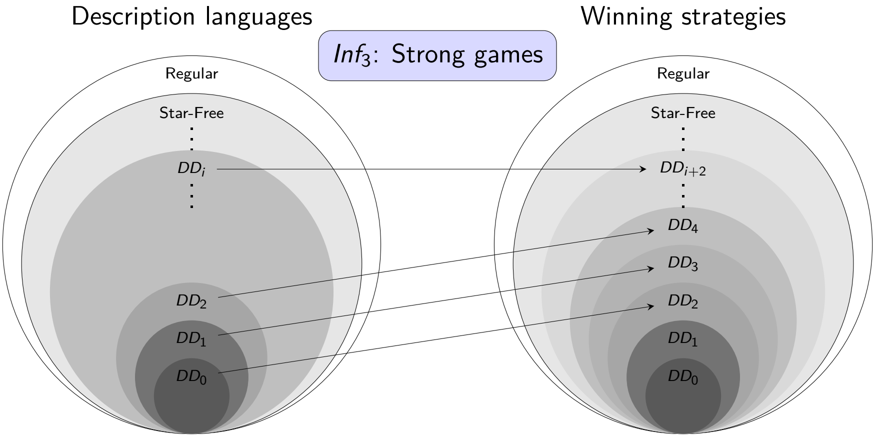

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