Languages vs. ω-Languages in Regular Infinite Games

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Infinite games

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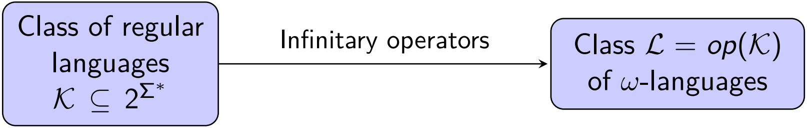
Winning condition: $L \subseteq \Sigma^\omega$. Winning strategies: $K_1, K_2 \subseteq \Sigma^*$

Languages and games

Languages and games

Class of regular languages
$\mathcal{K} \subseteq 2^{\Sigma^*}$

Languages and games



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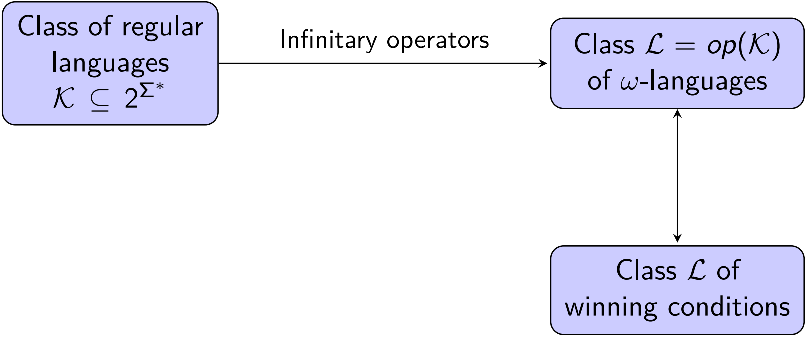

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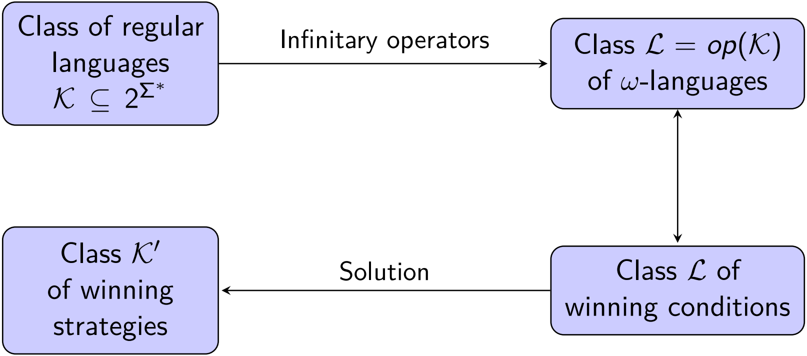

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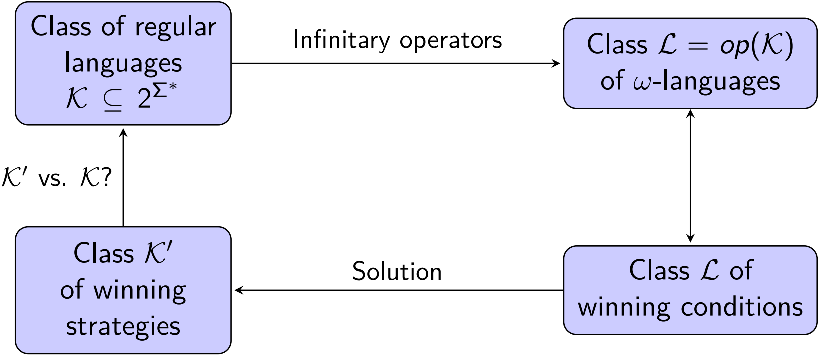

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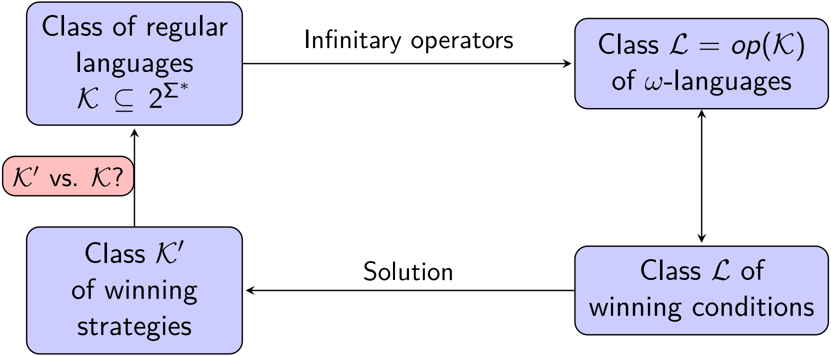

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Motivation and results

Do simple games have simple strategies?

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Description languages | Winning strategies

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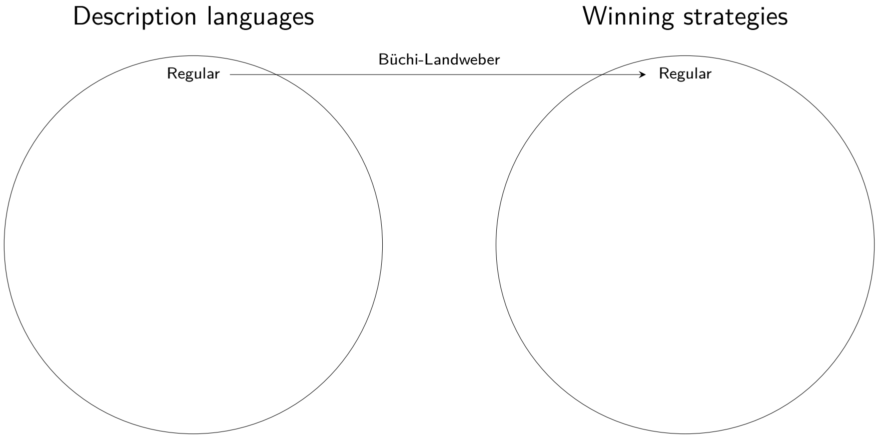

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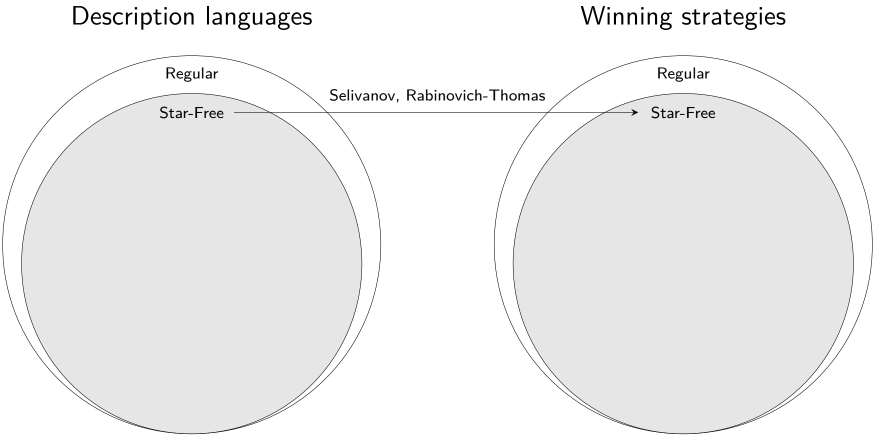

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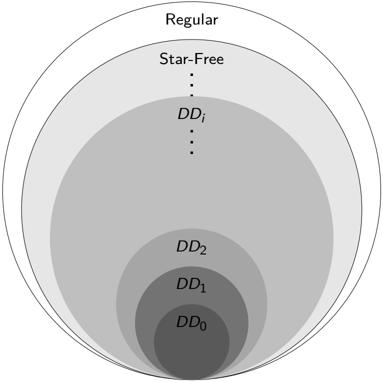
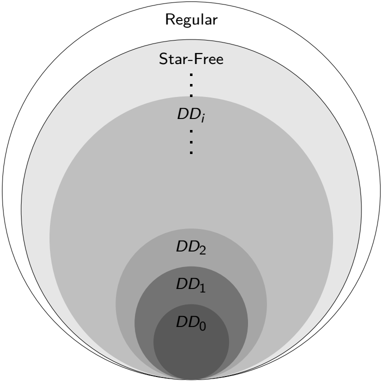


Winning strategies



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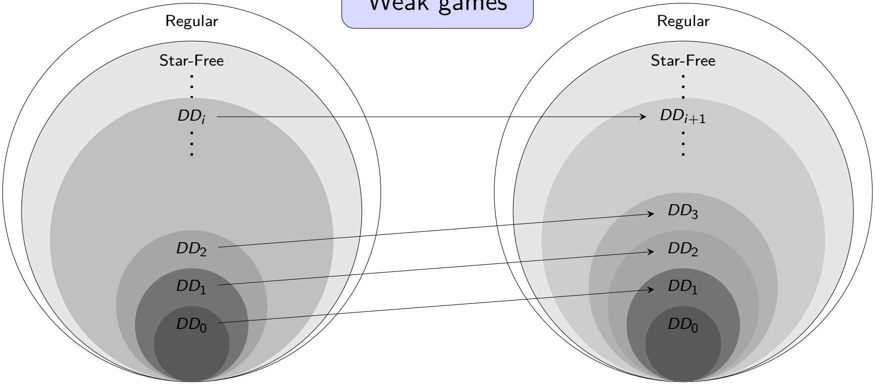
"Weak games"



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"Strong games"

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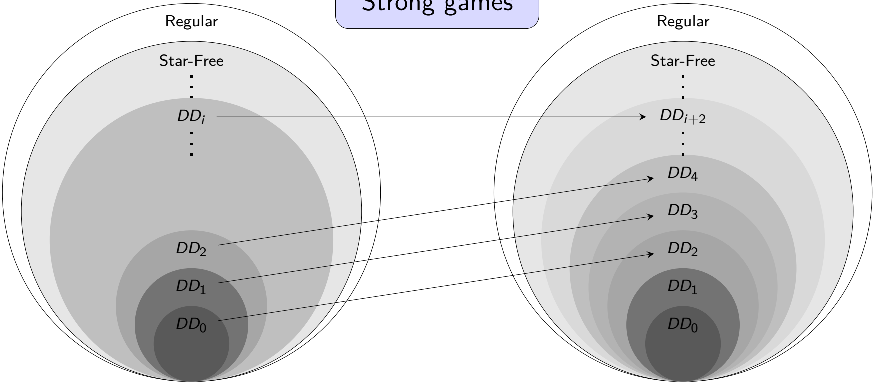

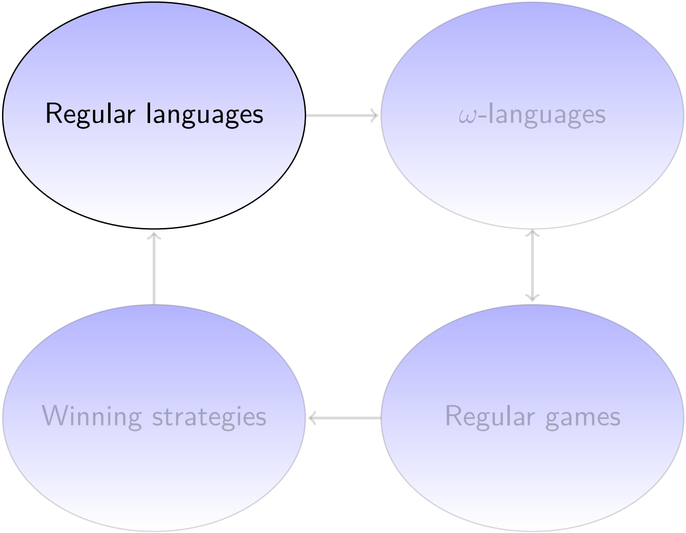
Regular languages
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Dot-depth hierarchy

Regular languages below SF

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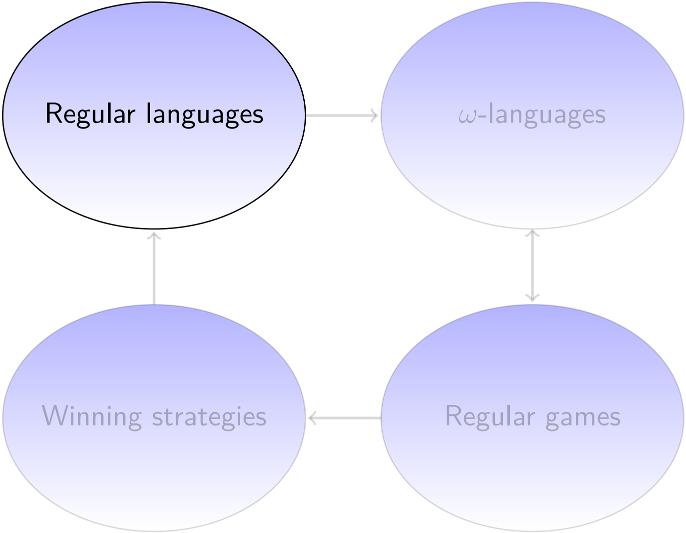
Strict hierarchy:

- $\mathrm{DD}_i \subsetneq \mathrm{DD}_{i+1}$
- $\bigcup_{i \in \mathbb{N}} \mathrm{DD}_i = \mathrm{SF}$ (star-free languages)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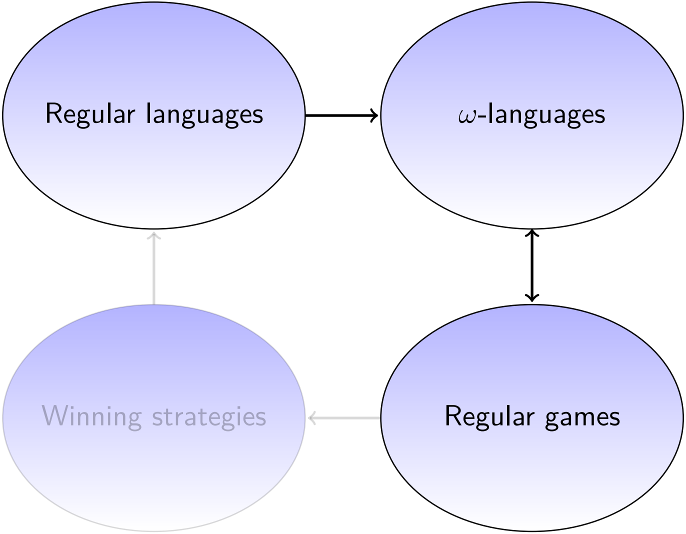
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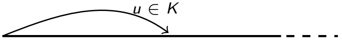
- $\text{ext}(K) \coloneqq \{\alpha \in \Sigma^\omega \mid \alpha =$ 

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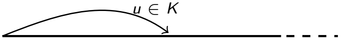
- $\text{ext}(K) \coloneqq \{\alpha \in \Sigma^\omega \mid \alpha =$ 

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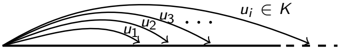
- $\lim(K) \coloneqq \{\alpha \in \Sigma^\omega \mid \exists^\omega i \in \mathbb{N}, \alpha =$ 

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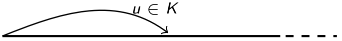
- $\mathrm{ext}(K) \coloneqq \{\alpha \in \Sigma^\omega \mid \alpha =$ 

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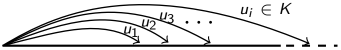
- $\lim(K) \coloneqq \{\alpha \in \Sigma^\omega \mid \exists^\omega i \in \mathbb{N}, \alpha =$ 

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For $\mathcal{K} \subseteq 2^{\Sigma^*}$:

- $\mathrm{ext}(\mathcal{K}) \coloneqq \{L \subseteq \Sigma^\omega \mid L = \mathrm{ext}(K), K \in \mathcal{K}\}$
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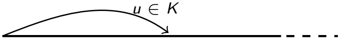
- $\text{ext}(K) := \{\alpha \in \Sigma^\omega \mid \alpha =$ 

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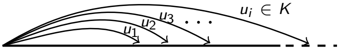
- $\lim(K) := \{\alpha \in \Sigma^\omega \mid \exists^\omega i \in \mathbb{N}, \alpha =$ 

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- $\text{BC}(\text{ext}(\mathcal{K})) := \{$Boolean combinations over $\text{ext}(\mathcal{K})\}$ (Weak games)
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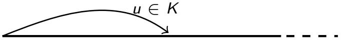
- $\mathrm{ext}(K) \coloneqq \{\alpha \in \Sigma^\omega \mid \alpha =$ 

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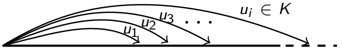
- $\lim(K) \coloneqq \{\alpha \in \Sigma^\omega \mid \exists^\omega i \in \mathbb{N}, \alpha =$ 

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Note: ω-REG $= \mathrm{BC}(\lim(\mathrm{REG}))$

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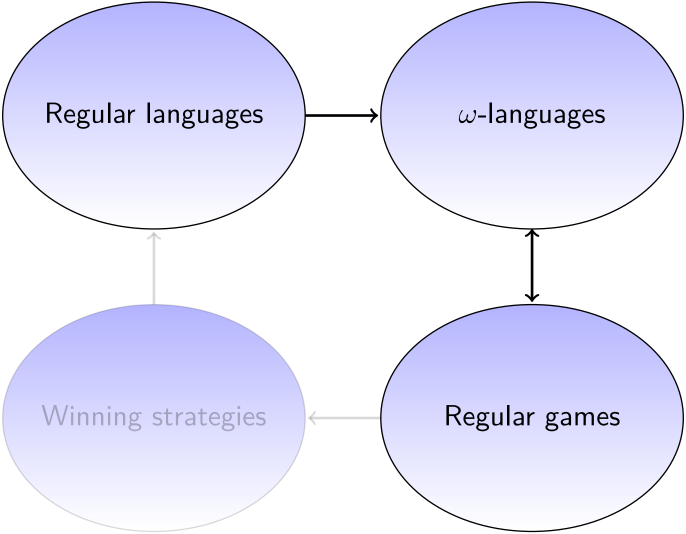
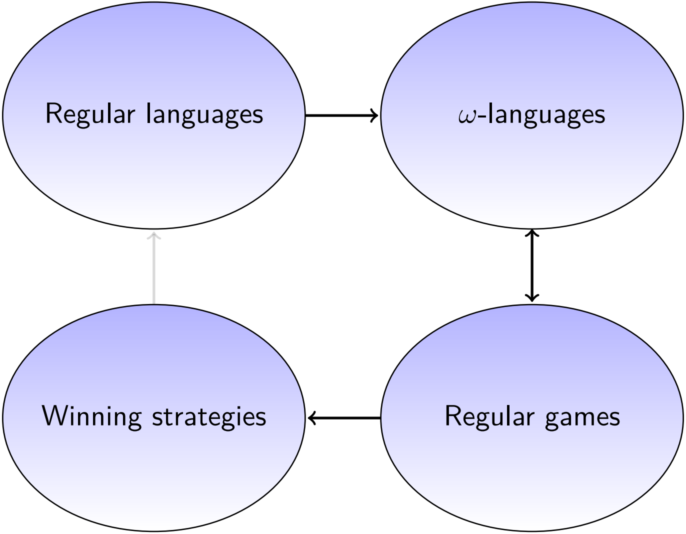

Regular languages
ω-languages
Winning strategies
Regular games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Games to regular languages

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Theorem (Büchi-Landweber)

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Theorem (Selivanov, Rabinovich-T.)

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The simple game: solution

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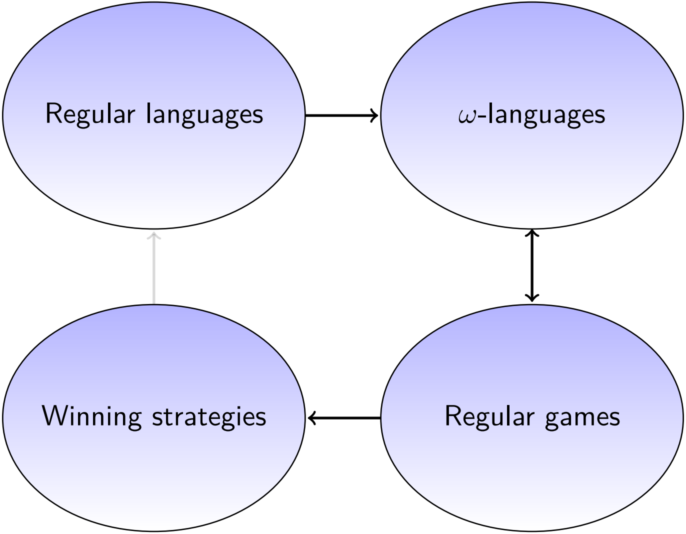
Game $L \in BC(\lim(DD_0))$. Strategy $K_0, K_1 \in DD_2 \setminus DD_1$


Regular languages
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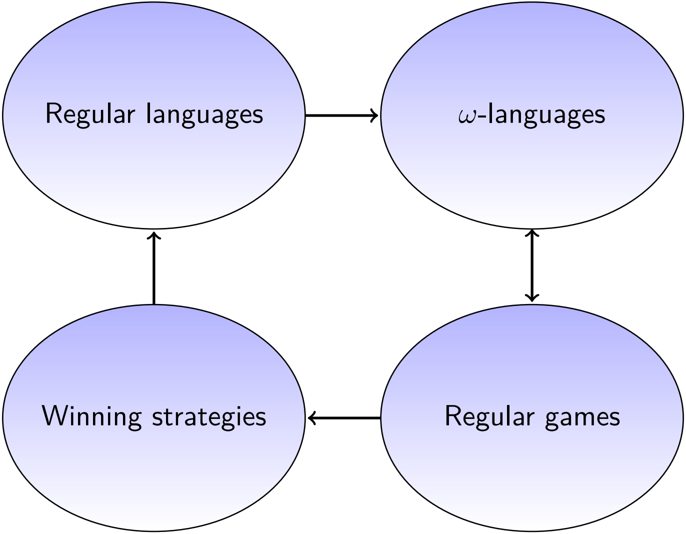
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Theorem (Weak games)

Games in $\mathsf{BC}(\mathsf{ext}(\mathsf{DD}_i))$ *are determined with winning strategies in* DD_{i+1}.

Proof.

□

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Theorem

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There are games in $\mathrm{BC}(\mathrm{ext}(\mathrm{DD}_1))$ *that do not admit* DD_1 *strategies.*

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- Player 2 has a winning strategy, $K \in DD_2 \setminus DD_1$

Conclusion

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PT	DD_2 but not DD_1	PT
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- Open: Do there exist games in $\mathrm{BC}(\lim(\mathrm{DD}_i))$ that do not admit any DD_{i+1} strategies?
- Open: How many states are needed for winning strategies?