Trees and Term Rewriting in 1910: On a Paper by Axel Thue *

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Abstract

Many of Axel Thue's ideas have been influential in theoretical computer science. In particular, Thue systems, semi-Thue systems and his work on the combinatorics of words are well-known. Here we consider his 1910 paper which contains many notions and ideas about trees, term rewriting and word problems which are surprisingly modern and have later come to play important roles in mathematics, logic, and computer science.

1 Prologue

The Norwegian mathematician Axel Thue (1863-1922) is well-known to theoretical computer scientists for his influential work on the combinatorics of words and word replacement systems. This is witnessed by the common use of terms such as "Thue-Morse sequence", "Thue system", and "semi-Thue system". There is, however, another thread of far-reaching ideas of Axel Thue which seems to be largely overlooked. These ideas appear in an innocent-looking paper of 1910 [Thue10] with the uninformative title *Die Lösung eines Spezialfalles eines generellen logischen Problems* ("The solution of a special case of a general logical problem"). In this article, Thue introduces trees and tree rewriting, poses the word problem for systems of tree identities, and - explicitly or implicitly - uses concepts such as the Church-Rosser property, confluence, termination, and even critical pairs, which now are of fundamental importance in term rewriting theory. Moreover, as remarked by Büchi [Büch89], a passage in this paper suggests that Thue surmised the possibility that a mathematical decision problem can be algorithmically unsolvable.

The purpose of the present note is to assess this remarkable paper from the point of view of contemporary theoretical computer science. After a brief sketch of Axel Thue's life, we describe the contents of the paper, emphasizing its conceptual contributions.

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2 Axel Thue: Life and Letters

Axel Thue was born in 1863 in Tönsberg, a town on the Oslo fjord. He studied mathematics in Oslo (the town was then still called Kristiania) and obtained his degree there in 1889. The years 1891-94 he enjoyed a research fellowship at the University of Oslo, but much of the years 1891-92 he spent in Berlin and Leipzig on a scholarship Sophus Lie had helped him to obtain. In Berlin he attended lectures by Kronecker and Helmholtz. In 1894 he became a senior lecturer at the technical college in Trondheim. From 1903 until his death in 1922 Thue held the professorship in applied mechanics at the University of Oslo. He was married to Lucie Collet Lund with whom he had seven children.

Thue's main field of research was number theory. His most famous work dealt with the approximation of algebraic numbers and led to what today is called the Thue-Siegel-Roth Theorem. Another prominent result known as "Thue's Theorem" belongs to the theory of Diophantine equations. Most of his papers appear in the volume *Selected Mathematical Papers of Axel Thue* [Thue77] which also contains a short biography written by Viggo Brun and an introduction by the number theorist Carl Ludwig Siegel.

Four out of the 35 papers in that volume deal with subjects which today would be counted as theoretical computer science. The paper *Über unendliche Zeichenreihen* of 1906 and the long article *Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen* of 1912 constitute a foundation of the subject of combinatorics on words, in particular of morphic infinite words (of which the "Thue-Morse word" is the most prominent example). This work has been discussed in detail by Berstel in [Bers92]. In the paper *Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln* (1914) which is, in fact, a continuation of the paper to be considered here, Thue introduces word rewriting systems and poses the word problem for them. This problem was shown to be undecidable by Emil Post in his paper *Recursive unsolvability of a problem of Thue* [Post47], where the terms "semi-Thue system" and "Thue system" were also introduced.

While these three papers of Thue were taken up after some time by others and led to the foundation of new subjects in discrete mathematics, the fourth paper, well hidden behind its empty title *Die Lösung eines Spezialfalles eines generellen logischen Problems*, remained rather unnoticed. The theories of term rewriting, tree languages, and types, all present in this paper, developed only decades later without reference to Thue. Probably it was Büchi who first put the paper into the context of theoretical computer science. In his posthumously published book *Finite Automata, Their Algebras and Grammars* [Büch89] one finds several pointers to Thue's work of 1910.

Let us look at the paper more closely.

3 Overview of the paper

The starting point of the paper is the idea of representing by a tree the way a composed *concept* is formed from some given *elementary concepts*. It may happen that two such representations are actually equivalent in view of some identities between trees given as axioms. The "logical problem" which Thue studies in this paper is the question whether it is possible to transform one such representation of a concept into another by successive steps justified by the axioms. The problem is essentially the word problem for terms and when stating and solving some special cases of it, Thue at least implicitly uses notions and tools which now are well-known in term rewriting theory.

These fundamental ideas are developed in the first main part of the paper, called "Abteilung A" (21 pages). In this part, Thue also makes very interesting observations on the solvability of certain special types of the transformability problem.

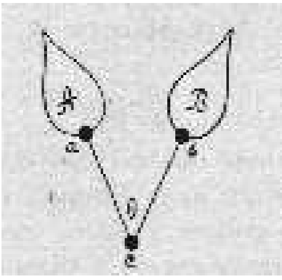
In the second main part, "Abteilung B" (16 pages), one of these special cases is treated in detail and solved. Since this part is rather technical, we confine our subsequent discussion to the conceptually more interesting first part. Our sections below correspond to Thue's division of "Abteilung A" into four sections.

4 Trees as representations of concepts

Thue starts his paper with the sentence (in our translation)

> *It may happen that from an arbitrary concept A of a category P and an arbitrary concept B of a category Q one can form by a certain procedure or operation θ in a unique way a concept C of a category R.*

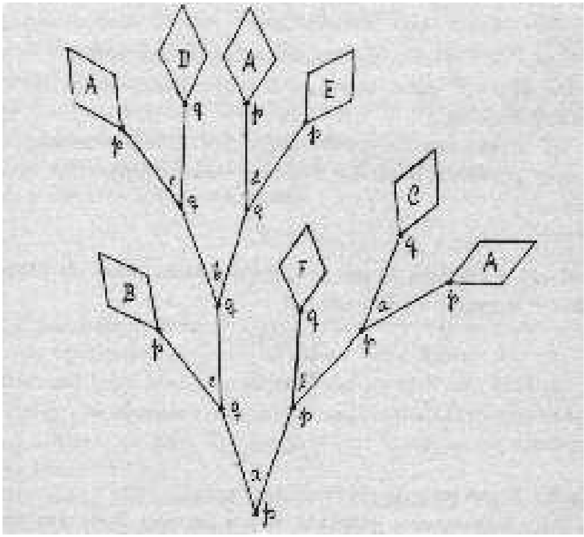
Thue illustrates the combination of two concepts A and B by an operation θ by a geometric figure consisting of a leaf a for A, a leaf b for B and a root-node c for the result of the operation.



Successive applications of such operations lead to ever bigger (binary) trees, each of them representing a concept. By the rules of concept formation, each tree belongs to a certain well-defined category ("Begriffskategorie"). Thue also remarks that the order of the operands may affect the result of the operation, and hence his trees are to be regarded as ordered. Using a more modern terminology, one may say that he considers elements in a many-sorted algebra which are representable by terms generated from constants by means of many-sorted functions. [1]

The process of evaluating the category of a given tree (i.e., the sort of a term) is illustrated in Thue's next figure. The notation is somewhat conflicting with the previous figure since now the letters a, b, c, etc. are used to denote operations, and the symbols p, q, etc. at the nodes record the categories (sorts) of the values of the subtrees rooted at the different nodes.

[1] In graph theory, where trees have been considered long before Thue, trees may be rooted, but they are usually unordered. It is remarkable that in his 1857 paper on the enumeration of trees, also Cayley arrives at the tree concept by considering operations and their compositions. However, he had in mind operations like differentiation and his tree representations would hardly have suggested Thue's problems. For an interesting account of the history of trees in graph theory we refer the reader to [BLW76].



One may view this figure as a representation of the run of a *tree automaton* which evaluates the sort of the given input tree. For example, the transition $c : (p, q) \mapsto q$, which appears here twice, indicates that applying the operation c to a concept of category p and a concept of category q yields a concept of category q. The evaluation takes place in a frontier-to-root manner, and hence by this example Thue antedates the kind of tree automata introduced in the 1960's by Thatcher and Wright, Doner and others. These authors had as their starting point the view of automata as algebras propagated by Büchi and Wright (cf. [Büch89] and [GeSt97] for references). In the subsequent discussion Thue drops this semantic aspect of category evaluation, and trees appear just as representations of terms.

At the end of the first section, Thue introduces some basic tree terminology, such as "erster Knotenpunkt" ("first node") for *root*, "Blatt" for *leaf*, "Faden", "Knotenlinie" for *path*, and "Zweig" ("twig", "branch") for *subtree* (including the cases of the given tree itself and its leaves). Furthermore, he notes the relationships $k = 2b - 1$, $o = k - b = b - 1$ between the total number k of nodes, the number b of leaves, and the number o of inner nodes of a binary tree.

5 The word problem

5.1 Thue's attempts to formulate the problem

In Section 2, Thue tries several formulations of the problem whether the equality of two trees can be inferred from some given axioms. He begins with the remark that two expressions s and t (trees) may denote the same concept, and this is expressed by writing $s = t$. No formal semantic definition of equality is presented, but later a proof-theoretic meaning is given: an equality $s = t$ holds if it can be syntactically derived from a set of equalities, the axioms in the given situation. However, before arriving at a clear formulation of the general word problem as it is now known, Thue goes through some intermediate stages which seem to suggest other problems. First he states that it may happen that $s = t$ whenever the *leaves* denote concepts which satisfy some equalities $p_1 = q_1, \dots, p_n = q_n$. At this point it seems that Thue is thinking of universally quantified equational implications of the form

$$(\forall x_1) \dots (\forall x_k)(p_1 = q_1 \wedge \dots \wedge p_n = q_n \to s = t),$$

where the variables x_i are the leaf symbols appearing in the trees concerned.

However, next he asks the question whether the equality $s = t$ can be derived assuming that the leaves of s and t satisfy the conditions $p_1 = q_1, \dots, p_n = q_n$ where the p_i, q_i are trees. The problem is immediately restated as the following *transformation problem*: Is it possible to prove an equality $s = t$ by successively transforming the terms into each other using the given axioms? Since at this point the leaf symbols are regarded as constants, this would be the word problem for ground equations, i.e., equations without variables. After mentioning an "important special case" which is not discussed any further (in which the left-hand side of each axiom is just a leaf which does not appear anywhere else in the system), Thue restates the problem without any change. However, at this point he also decides to simplify matters by limiting himself to the one-sorted case; he does not return to the issue of different "Begriffskategorien".

When the problem is then stated once more, for the third time within two pages, Thue declares explicitly that a leaf symbol does not represent a fixed concept. In other words, leaf symbols are regarded as variables and the axioms $p_1 = q_1, \dots, p_n = q_n$ are now assumed to be pairs of "tree-like figures" such that equality holds whatever concepts the leaves are replaced by so that the same leaf (symbol) is replaced everywhere by the same concept. Now the question posed is clearly the *Word Problem* which, stated in a modern terminology, asks whether we can decide whether an identity

$$s(x_1, \dots, x_n) = t(x_1, \dots, x_n)$$

holds for all values of the variables $x_1, \dots, x_n$ assuming that a given finite set Γ of identities, the *axioms*,

$$p_1(x_1, \dots, x_n) = q_1(x_1, \dots, x_n), \ \dots \ , p_k(x_1, \dots, x_n) = q_k(x_1, \dots, x_n)$$

hold for all values of the variables $x_1, \dots, x_n$.

The question could concern either the *logical consequence relation* $\Gamma \models s = t$ or the *derivability relation* $\Gamma \vdash s = t$, where Γ denotes the set of axioms. However, Thue had neither a definition of algebras (models) nor of the logical consequence relation, and later it becomes again clear that he is thinking of derivability. Of course, the distinction between the two was made much later and the equivalence of $\Gamma \models s = t$ and $\Gamma \vdash s = t$ was established in 1930 by Gödel's Completeness Theorem for first-order logic, and then explicitly for identities in 1935 by Birkhoff in his Completeness Theorem for Equational Logic. It is also to be noted that Birkhoff's proof system (cf. [BuSa81], for example) essentially formalizes the rules used by Thue.

The fact that Thue did not distinguish between constants and variables may be one reason why he did not arrive at a clear formulation of the word problem more directly. Originally the leaf symbols appear as constants representing elementary concepts, but in the identities (and later in the rewrite rules) they are used as variables which may assume any trees as their values.

The examination of the word problem is interrupted by a new discussion of the tree concept. First Thue defines inductively the set of *unlabelled (binary) trees*, which are called 'branching diagrams' ("Verzweigungsfigur" or just "Verzweigung"). A *labelled tree* ("Zeichenbaum") is then obtained by adding a mapping which assigns labels to the nodes of an unlabelled tree. Two sets of symbols are assumed, one for labelling leaves and another for the inner nodes. As it later turns out, the leaf symbols are treated as variables while the labels of the inner nodes can be regarded as binary function symbols. The definition is then very similar to the now common definition (cf. [GeSt97], for example) of a tree as a mapping from a *tree domain* to the union of a set X of leaf symbols and a ranked alphabet Σ of operation symbols, the main difference being that Thue's tree domains are geometric figures rather than sets of node addresses. Thue defines recursively what it means that two unlabelled trees are *identical*, and then he defines the identity of two labelled trees.

When Thue turns again to the word problem, the novelty is a clear definition of the *tree replacement* operation. Assuming again a given finite set

$$\Gamma = \{p_1 = q_1, \ldots, p_k = q_k\}$$

of (universally quantified) tree identities, the *axioms*, Thue calls two trees s and t *congruent*, and writes $s \sim t$, if s contains an instance of one side of an axiom as a subtree and t is obtained when this subtree is replaced by the corresponding instance of the other side of that axiom; an *instance* of a tree is obtained by applying to the tree a *substitution* which replaces each leaf by a tree so that identical leaves are replaced by identical trees. When an axiom is used this way, the same substitution is applied to both of its sides. (Thue does not speak about substitutions and instances, but his intentions are perfectly clear.) Next, two trees s and t are said to be *equal* if there are trees $r_1, \ldots, r_m$ such that

$$s \sim r_1 \sim r_2 \sim \ldots r_m \sim t.$$

Thue writes this simply $s = t$, but the relation defined is actually what now is called the *Thue congruence* generated by the set Γ and denoted by $\equiv_\Gamma$, for example. When trees are replaced by terms, the relation $\equiv_\Gamma$ is the *deductive closure* of Γ, and by Birkhoff's Theorem it is also the set of identities which hold in all models of Γ. Moreover, for any set Γ of term identities, the relation $\equiv_\Gamma$ can also be described as the *fully invariant congruence* on the term algebra generated by Γ (cf. [BuSa81], for example). Such relations, and their variations obtained in different contexts, are the main object of interest of equational logic. An *instance* of the word problem simply asks whether a given identity (when viewed as a pair of terms, trees, strings, etc.) belongs to $\equiv_\Gamma$ for a given Γ.

5.2 The question of solvability and some solvable cases

The word problem is then stated again in its final form, now clearly as a question of derivability. At this point Thue makes the intriguing remark

> *Eine Lösung dieser Aufgabe im allgemeinsten Falle dürfte vielleicht mit unüberwindlichen Schwierigkeiten verbunden sein*

that is to say,

> *A solution of this problem in the most general case may perhaps be connected with insurmountable difficulties.*

Thus it is conceivable that Thue envisaged the possibility that a decision problem is unsolvable, moreover, in the case of a concrete problem which today is known to be really unsolvable. Thue's comment contrasts remarkably with common beliefs of his contemporaries. For example, the statement of Hilbert's 10th problem [Hilb01] asks simply for an algorithm for deciding whether a system of Diophantine equations has solutions, and the possibility of undecidability is not even mentioned.

After his remark on "insurmountable difficulties", Thue mentions some cases in which the problem is solvable. First of all, if both sides of every axiom contain the same number of occurrences of each variable, then congruent trees always have the same number of leaves, and hence the length of a non-looping derivation starting from a given tree is effectively bounded. As a second type of trivially solvable problems, Thue mentions the case where in every axiom the maximum heights at which a variable appears coincide for both sides. Again, this allows only a bounded number of trees congruent with any given tree.

The further cases which Thue announces to be solvable are nontrivial. In all of them there is just one variable and one binary operation symbol. Hence they actually concern unlabelled trees in which all leaves are the same variable. Since they are the only type of tree considered in the rest of the paper, we shall henceforth call them simply "trees".

6 Towards term rewriting

6.1 From identities to rewriting rules

Section 3 begins with a formula for the number $R(n)$ of unlabelled binary trees with n leaves. Next Thue defines the *concatenation* $s(t)$ of two trees s and t as the tree obtained by replacing every leaf of s by the tree t. A *prime tree* is defined as a tree which cannot be represented as the concatenation of two non-trivial trees. Thue then notes without proof that every tree can be represented in a unique way as a concatenation of prime trees. Although the topic is not pursued any further, this passage could be of some historical interest from the point of view of the theory of *tree codes*.

After these digressions, the word problem is restated for the case at hand. However, now Thue also states explicitly that any instance $p(r) = q(r)$ of an axiom $p = q$ obtained by substituting a tree r for the leaves, is also regarded as an axiom.

The subsequent discussion concerning this case of the word problem involves some of the most interesting thoughts of the paper. Although in the present context 'tree' always means a tree in which all leaves are labelled with the same variable and all inner nodes with the same binary symbol, the general ideas are not restricted to this case.

Let $b(t)$ denote the number of leaves of a tree t. Since $2b(t) - 1$ is the total number of nodes, $b(t)$ is also a measure of the size of t. Thue considers now a system of identities

$$\Gamma \quad = \quad \{p_1 = q_1, \ldots, p_k = q_k\}$$

satisfying the following three conditions:

(C1) $b(p_i) \geq b(q_i)$ for every $i = 1 \ldots, k$;

(C2) if $b(p_i) > b(q_i)$ and $b(p_j) > b(q_j)$ where $1 \leq i \neq j \leq k$, then p_i is not an instance of p_j, i.e., there is no tree r such that $p_i = p_j(r)$;

(C3) if $b(p_i) > b(q_i)$ $(1 \leq i \leq k)$, then no proper non-leaf subtree p of p_i is unifiable with any p_j $(1 \leq j \leq k)$, that is to say, there are no trees r and s such that $p(r) = p_j(s)$.

Thue's formulation of condition (C3) actually forbids only the unifiability of any left-hand side with its own proper non-leaf subtrees, but this is obviously not his intention. Moreover, later he uses condition (C1) in a stronger form assuming that $b(p_i) > b(q_i)$ for all $i = 1 \ldots, k$. In any case, these conditions show clearly that Thue intends to regard the equations as oriented rewrite rules $p_i \rightarrow q_i$ which are always used so that an instance $p_i(r)$ of the left-hand side is replaced by the corresponding instance $q_i(r)$ of the right-hand side.

6.2 Some modern terminology

The above conditions (C1) - (C3) can be better appreciated if we first recall some definitions and facts about rewriting systems in a modern terminology.

As we shall explain below, a rewriting system $R = \{l_1 \to r_1, \ldots, l_m \to r_m\}$ gives an algorithm for solving all instances of the word problem involving a given set of identities $\Gamma = \{p_1 = q_i, \ldots, p_k = q_k\}$ if it is *complete* and *equivalent* to Γ. This holds regardless whether we are speaking about trees, terms, strings or graphs. For the sake of definiteness, let us assume that the objects are Σ-terms with variables in X, where Σ is a set of function symbols and X a set of variables. Thue's special trees are essentially pictures of such terms when Σ consists of one binary symbol and X is a one-variable set. The rewrite relation $\Rightarrow_R$ on the set of terms is defined as follows: $s \Rightarrow_R t$ if t is obtained from s by replacing an instance $\sigma(l_i)$ of a left-hand side of a rule by the corresponding instance $\sigma(r_i)$ of the right-hand side; here σ is a substitution which replaces each variable by a term. Let $\Rightarrow_R^*$ be the reflexive transitive closure of $\Rightarrow_R$, i.e., $s \Rightarrow_R^* t$ holds if and only if $s = t$ or there is a *reduction sequence*

$$s \Rightarrow_R t_1 \Rightarrow_R \ldots \Rightarrow_R t_{n-1} \Rightarrow_R t$$

which converts s in $n \geq 1$ steps into t. The converses of $\Rightarrow_R$ and $\Rightarrow_R^*$ are denoted by $\Leftarrow_R$ and $\Leftarrow_R^*$, respectively. Moreover, let $\Leftrightarrow_R^*$ be the equivalence generated by $\Rightarrow_R$.

A rewriting system R is called

1. *terminating* if no infinite reduction sequence $t_1 \Rightarrow_R t_2 \Rightarrow_R \ldots$ exists;

2. *confluent* if any two descendants of the same term have a common descendant: if $u \Leftarrow_R^* s \Rightarrow_R^* v$ for some terms s, u, and v, then $u \Rightarrow_R^* t \Leftarrow_R^* v$ for some t;

3. *complete* if it is terminating and confluent.

For a complete rewriting system every term t has a unique *normal form* $t\downarrow$, i.e., a descendant of t which cannot be reduced any further by R. That every term can be reduced to an irreducible term follows by termination, and that there cannot be two distinct irreducible descendants follows from the confluence of R. Moreover, as shown by Newman [Newm42], any confluent rewriting system R has the *Church-Rosser property*: if $s \Leftrightarrow_R^* t$, then there exists a term u such that $s \Rightarrow_R^* u \Leftarrow_R^* t$. For systematic presentations of these matters we refer the reader to [Huet80], the survey [DeJo90], or the books [Aven95] and [BaNi98].

A rewriting system R and a set Γ of identities are said to be *equivalent* if $\Leftrightarrow_R^* = \equiv_\Gamma$. A complete rewriting system R equivalent to a system Γ of identities gives an algorithm for solving the word problem for Γ. Indeed, for any terms s and t, $s \equiv_\Gamma t$ if and only if $s\downarrow = t\downarrow$, and the normal forms $s\downarrow$ and $t\downarrow$ can be obtained by reducing the terms s and t by the rules of R in an arbitrary manner until irreducible terms are obtained.

6.3 Some fundamental insights

Thue converts his system of tree identities $\Gamma = \{p_1 = q_1, \ldots, p_k = q_k\}$ to an equivalent rewriting system $R = \{p_1 \to q_1, \ldots, p_k \to q_k\}$ simply by orienting the identities. As noted above, he actually uses condition (C1) in a stronger form which requires that $b(p_i) > b(q_i)$ for every $i = 1, \ldots, k$. This implies immediately that R is terminating since $b(s) > b(t)$ whenever $s \Rightarrow_R t$. Moreover, when (C1) is interpreted this way, conditions (C2) and (C3) apply to all rules, and Thue uses this fact to show that R is also confluent. Seen from our modern vantage point, this result is obvious since now (C2) and (C3) rule out all *critical pairs*. A critical pair arises when the left-hand side of some rule can be unified with a non-leaf subtree of the left-hand side of a rule. The significance of critical pairs was explicitly noted by D.E. Knuth and P.B. Bendix in their landmark paper [KnBe70]. They showed that for confluence it is necessary that every critical pair converges, i.e., has a common descendant, and for a terminating system this is also sufficient. In their famous *completion method* a complete rewriting system is sought by orienting, if possible without loosing termination, non-converging critical pairs into new rules. Since conditions (C2) and (C3) forbid critical pairs, Thue's systems are complete. If (C2) and (C3) are combined, we may say that Thue proved the following result:

Proposition 6.1. *If $R = \{p_1 \to q_1, \ldots, p_k \to q_k\}$ is a tree rewriting system (over unlabelled binary trees) such that*

1. *$b(p_i) > b(q_i)$ for every $i = 1, \ldots, k$, and*
2. *no left-hand side p_i is unifiable with any non-leaf subtree of a left-hand side of another rule or with a proper non-leaf subtree of itself,*

then every tree can be reduced to a unique normal form.

Of course, the result follows immediately from what we now know about complete rewriting systems, but it is interesting that Thue explicitly introduces conditions which imply the two properties needed for completeness.

It can also be noted that in the proof Thue uses an idea which lies at the heart of the technique known as *proof transformation* [Bach91]. He shows that in any proof

$$(*) \qquad s \sim \ldots \sim u \sim v \sim w \sim \ldots \sim t$$

of $s \Leftrightarrow_R^* t$, where each $\sim$ represents either $\Rightarrow_R$ or $\Leftarrow_R$, every *peak*

$$u \Leftarrow_R v \Rightarrow_R w$$

can be eliminated, and that the proof $(*)$ thus can be transformed into a *reduction proof* $s \Rightarrow_R^* r \Leftarrow_R^* t$. Hence Thue actually shows that any R satisfying the conditions of Proposition 6.1 has the Church-Rosser property. Strangely enough, he does not note that this implies the solvability of the word problem of the given system of equations.

7 Additional results

The concluding section of "Abteilung A" contains some supplementary remarks and results about the word problem. Here Thue also starts labelling his results as theorems ("Satz 1", "Satz 2", etc.[2]), so it seems that he regards them as the first significant steps towards the main goals of the paper. Let us review briefly the points of methodological interest.

7.1 Proofs in normal form

The first result ("Satz 1") concerns a rather special type of systems of identities, but it is worth mentioning since it says that identities derivable by such systems have proofs of a certain normal form. Let Γ consist of just one tree identity $p = q$, where q is a one-node tree. (All trees are again unlabelled and binary as described above.) Let $R = \{p \to q\}$ be the tree rewriting system obtained by orienting the only identity. Thue's "Satz 1" says that if $s \equiv_\Gamma t$ for some trees s and t, then there exists a tree r such that $s \Leftarrow_R^* r \Rightarrow_R^* t$. That every identity derivable from Γ has such a normal form proof is shown by induction on the length of the given proof

$$s \sim s_1 \sim \ldots \sim s_{m-1} \sim t.$$

7.2 Local confluence and the Church-Rosser property

The second theorem of Section 4, "Satz 2", makes use of one more basic idea of rewriting theory. A rewriting system R is *locally confluent* if $u \Leftarrow_R s \Rightarrow_R v$ always implies the existence of a t such that $u \Rightarrow_R^* t \Leftarrow_R^* v$. Confluence obviously implies local confluence but for terminating systems also the converse holds [Newm42]. This result, known as Newman's Lemma and actually true for abstract reduction systems, is important since verifying local confluence is a finite task usually much easier than proving confluence directly. Thue's "Satz 2", given without proof, can now be stated as follows.

Proposition 7.1. *If $R = \{p_1 \to q_1, \ldots, p_k \to q_k\}$ is a locally confluent tree rewriting system such that $b(p_i) > b(q_i)$ for every $i = 1, \ldots, k$, then every tree has a unique normal form.*

Of course, the assumption $b(p_i) > b(q_i)$ $(i = 1, \ldots, k)$ implies termination. That local confluence and termination together mean completeness is confirmed by the third theorem: in "Satz 3" Thue states that under these assumptions any two equivalent trees can be transformed to a unique common irreducible form, which means, in particular, that such a system has the Church-Rosser property.

[2] In fact, Thue uses "Satz" for auxiliary statements while the two main results appearing in "Abteilung B" are called "Theorem I" and "Theorem II", respectively.

The sufficiency of local confluence is utilized again in the proof of the fourth theorem ("Satz 4") which gives an example of an infinite tree rewrite system with unique normal forms.

8 Conclusion: The "special case"

The special case of the general logical problem to which the title of Thue's paper refers, is treated in the second part of the paper, "Abteilung B". Here Thue shows that the word problem is decidable for any system consisting of one tree identity which is of the form $p(q) = r$, where p and q are prime trees and r is the one-node tree. The proof is long, about fifteen pages, and rather involved. It is therefore understandable that Thue regards it as the main contribution of the paper. Today the general conceptual and methodological insights to be found in the first part of the paper appear more remarkable and astonishing.

It is a matter of speculation whether an earlier recognition of Thue's ideas would have led to a different development of the theory of term rewriting. At least a more informative title would have given the paper a better chance of gaining the influence which it had deserved.

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References

[Aven95] J. Avenhaus, *Reduktionssysteme*, Springer-Verlag, Berlin 1995.

[Bach91] L. Bachmair, *Canonical Equational Proofs*, Birkhäuser, Boston 1991.

[BaNi98] F. Baader and T. Nipkow, *Term Rewriting and All That*, Cambridge University Press, Cambridge 1998.

[Bers92] J. Berstel, Axel Thue's work on repetitions in words, In: P. Leroux and C. Reutenauer, eds., *Séries formelles et combinatoire algébrique*, Publications du LaCIM, Université du Quebec á Montreal, Montreal 1992, pp. 65-80.

[BLW76] N.L. Biggs, E.K. Lloyd and R.J. Wilson, *Graph Theory 1736-1936*, Clarendon Press, Oxford 1976.

[Büch89] J.R. Büchi, *Finite Automata, Their Algebras and Grammars* (ed. D. Siefkes), Springer-Verlag, New York 1989.

[BuSa81] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York 1981.

[DeJo90] N. Dershowitz and J.-P. Jouannaud, Rewrite systems, in: J. van Leeuwen (ed.) *Handbook of Theoretical Computer Science, Vol. B*, Elsevier Science Publisher B.V., Amsterdam 1990, pp. 243-320.

[GeSt97] F. Gécseg and M. Steinby, Tree languages, in: G. Rozenberg and A. Salomaa (eds.), *Handbook of Formal Languages, Vol. 3*, Springer-Verlag, Berlin 1997, pp. 1-87.

[Hilb01] D. Hilbert, Mathematische Probleme, *Arch. der Math. und Phys.* 1 (1901), pp. 44-63, 213-237.

[Huet80] G. Huet, Confluent reductions: Abstract properties and applications to term rewriting, *Journal of the ACM* 27, 1980, pp. 797-821.

[KnBe70] D.E. Knuth and P.B. Bendix, Simple word problems in universal algebra, in: J. Leech (ed.), *Computational problems in abstract algebra*, Pergamon Press, London 1970, pp. 263-297.

[Newm42] M.H.A. Newman, On theories with a combinatorial definition of 'equivalence', *Annals of Mathematics* 43 (2), 1942, pp. 223-243.

[Post47] E.L. Post, Recursive unsolvability of a problem of Thue, *J. Symb. Logic* 12 (1), 1-11, reprinted in: M. Davis (Ed.), *The Collected Works of Emil L. Post*, Birkhäuser, Boston 1994, pp. 503-513.

[Thue10] A. Thue, Die Lösung eines Spezialfalles eines generellen logischen Problems, *Kra. Videnskabs-Selskabets Skrifter. I. Mat. Nat. Kl. 1910, No. 8*, Kristiania 1910. Reprinted in [Thue77], pp. 273-310.

[Thue77] T. Nagel, A. Selberg, S. Selberg and K. Thalberg (eds.), *Selected Mathematical Papers of Axel Thue*. Universitetsforlaget, Oslo 1977.

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