Logical Theories and Compatible Operations

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Abstract

We survey operations on (possibly infinite) relational structures that are *compatible* with logical theories in the sense that, if we apply the operation to given structures then we can compute the theory of the resulting structure from the theories of the arguments (the logics under consideration for the result and the arguments might differ).

Besides general compatibility results for these operations we also present several results on restricted classes of structures, and their use for obtaining classes of infinite structures with decidable theories.

1 Introduction

The aim of this article is to give a survey of operations that can be performed on relational structures while preserving decidability of theories. We mainly consider first-order logic (FO), monadic second-order logic (MSO), and guarded second-order logic (GSO, also called MS_2 by Courcelle). For example, we might be interested in an operation f that takes a single structure $\mathfrak{a}$ and produces a new structure $f(\mathfrak{a})$ such that the FO-theory of $f(\mathfrak{a})$ can be effectively computed from the MSO-theory of $\mathfrak{a}$ (we call such operations (MSO, FO)-*compatible*), i.e., for each FO-formula φ over $f(\mathfrak{a})$ we can construct an MSO-formula φ^* such that

$$f(\mathfrak{a}) \models \varphi \quad \text{iff} \quad \mathfrak{a} \models \varphi^* .$$

The main application of such operations is to transfer decidability results for logical theories. This technique can be applied for single structures, as well as uniformly over classes of structures. The first approach is often used for infinite structures, but it becomes trivial if the structure is finite since each finite structure has a decidable MSO-theory (even a decidable full second-order theory). The second approach is also useful for classes of finite structures as not every such class has a decidable theory.

In order to process structures by algorithmic means, a finite encoding of the structure is required. Such encodings are trivial when structures are finite (though one may be interested into finding compact presentations), but the choice of encoding becomes a real issue when dealing with infinite structures. The approach using operations compatible with logical theories is as follows. Starting from a (countable) set B of structures all of which have a decidable theory for a certain logic L, we can construct new structures with a decidable theory (possibly for a different logic L') by applying operations from a fixed (countable) set O of operations of the above form. This gives rise to the class C of all structures that can be obtained from the basic structures in B by application of the operations in O. Every element of C can be represented by a term over $O \cup B$. Evaluating an L'-formula over a structure in C then amounts to constructing and evaluating L-formulae over structures from B.

Given such a definition of a class of structures, an interesting problem is to understand what structures can be encoded in this way and to give alternative characterisations of them. Before we give examples of such classes, let us briefly summarise the main operations we are interested in.

Interpretations. An interpretation uses logical formulae with free variables to describe relations of a new structure inside a given one. Each formula with n free variables defines the relation of arity n that contains all tuples satisfying the formula. Usually, the free variables are first-order variables and the universe of the new structure is a subset of the universe of the original structure. Depending on the type of the formulae one speaks of FO- and MSO-interpretations, and it is not difficult to see that these types of interpretations preserve the respective logic. We will frequently combine other operations with interpretations that perform some pre-processing and post-processing of structures.

Products. The simplest form is the *direct* or *Cartesian product* of two or more structures. A generalised version allows us to additionally define new relations on the product by evaluating formulae on the factors and relating the results on the different factors by another formula. Feferman and Vaught [FV59] proved that the first-order theory of such a product is determined by the first-order theories of its factors (see also [Mak04] for an overview).

Sums. To transfer the results of Feferman and Vaught for products to monadic second-order logic, Shelah considered sums (or unions) of structures instead of products [She75].

Iteration. The iteration of a structure consists of copies of the original structure that are arranged in a tree-like fashion. A theorem of Muchnik that has been proven in [Wal96, Wal02] states that the MSO-theory of an iteration can be reduced to the MSO-theory of the original structure.

Incidence Structure. The universe of the incidence structure contains, in addition to the elements of the original structure, all tuples that appear in some relation. This construction can be used to reduce the GSO-theory of a structure to the MSO-theory of its incidence structure [GHO02].

Power set. The power set of a structure consists of all of its subsets. The relations are transferred to the singleton sets and the signature additionally contains the subset relation. There is also a weak variant of the power-set operation that takes only the finite subsets of a structure. These constructions allow us to translate FO-formulae over the power-set structure to MSO-formulae over the original structure, and to weak MSO-formulae in case of finite sets [CL07b].

Of course, these operations can also be combined to obtain more complex ones. For example, applying a product with a finite structure followed by an MSO-interpretation yields a *parameterless* MSO-*transduction* (see e.g., [Cou94]). Or applying the power-set operation followed by an FO-interpretation gives an operation called a *set interpretation* (or finite set interpretation in the case of the weak power set) [CL07b].

Besides the general results on the compatibility of these operations, we are interested in their behaviour on special classes of structures. In particular we consider the following families.

Tree-interpretable structures are structures that can be obtained by the application of an interpretation to a tree. Here, the interpretation can be chosen to be first-order, weak monadic-second order, or monadic second-order without affecting the definition (if the tree is changed accordingly). This class coincides with the class of structures of finite partition width [Blu06]. The corresponding class of graphs consists of those with finite clique width [Cou04]. Seese [See91] conjectures that all structures with decidable MSO-theory are tree-interpretable.

Structures of finite tree width resemble trees. They can be characterised as the structures with a tree-interpretable incidence graph. A theorem of Seese [See91] states that all structures with decidable GSO-theory are have finite tree width.

Uniformly sparse structures are the structures where the relations contain “few” tuples. Over these structures the expressive powers of GSO and MSO coincide [Cou03]. A tree-interpretable structure is uniformly sparse if and only if it has finite tree width.

Structures FO*-interpretable in the weak power set of a tree* have a FO-theory which is reducible to the WMSO-theory of the tree. Special techniques are developed to study those structures. In particular, we present reductions to questions about WMSO-interpretability in trees.

Finally, we employ compatible operations to define classes of infinite structures with decidable theories. We use the following classes of structures to illustrate this method.

Prefix-recognisable structures. The original definition of this class is based on term rewriting systems [Cau96]. In our framework, these are all structures that can be obtained from the infinite binary tree by an MSO-interpretation, or equivalently by an FO-interpretation [Col07]. As the infinite binary tree has a decidable MSO-theory [Rab69], the same holds for all prefix-recognisable structures. A fourth definition can be given in terms of the configuration graphs of pushdown automata [MS85]. A graph is prefix-recognisable if and only if it can be obtained from such a configuration graph by factoring out ε-transitions. The class of HR-*equational structures* is a proper subclass of the prefix-recognisable structures [Cou89]. By definition, each prefix-recognisable structure is tree-interpretable and it is HR-equational if and only if it has finite tree width or, equivalently, if it is uniformly sparse.

The Caucal hierarchy. This hierarchy is defined by combining MSO-interpretations with the iteration operation. Starting from the set of all finite structures one alternatingly applies these two operations [Cau02]. The first level of this strict hierarchy corresponds to the class of prefix-recognisable structures. As both operations are compatible with MSO, one obtains a large class of infinite graphs with decidable MSO-theories. Each structure in the Caucal hierarchy is tree-interpretable.

Automatic structures. According to the original definition, the universe of an automatic structure is a regular set of words and the relations are defined by finite automata that read tuples of words in a synchronous way [Hod83]. In the same way one can define tree-automatic structures using tree automata instead of word automata (and an appropriate definition of automata reading tuples of trees).

In our approach, automatic structures are obtained via an FO-interpretation from the weak power set of the structure $\langle \omega, < \rangle$ (the natural numbers with order). In the same way, tree-automatic structures can be obtained from the infinite binary tree [CL07b]. By the choice of the operations it follows that each (tree-)automatic structure has a decidable FO-theory.

Tree-automatic hierarchy. Combining the previous ideas, one can consider the hierarchy of structures obtained by applying the weak power-set operation followed by an FO-interpretation to all trees in the Caucal hierarchy. It can be shown that this yields a strict hierarchy of structures with a decidable FO-theory.

The article is structured as follows. In the next section we introduce basic terminology and definitions. Section 3 is devoted to the presentation of the operations and basic results concerning their compatibility. Further results that can be obtained on restricted classes of structures are presented in Section 4. The use of compatible operations for defining classes of structures with decidable theories is illustrated in Section 5.

2 Preliminaries

Let us fix notation. We define $[n] := \{0, \dots, n-1\}$, and $\mathcal{P}(X)$ denotes the power set of X. Tuples $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle \in A^n$ will be identified with functions $[n] \to A$. We will only consider relational structures $\mathfrak{a} = \langle A, R_0, \dots, R_{n-1} \rangle$ with finitely many relations $R_0, \dots, R_{n-1}$ and where the universe A is at most countable.

An important special case of structures are trees. Let D be a set. We denote by D^* the set of all finite sequences of elements of D. The empty sequence is $\langle\rangle$. The *prefix ordering* is the relation $\preceq \subseteq D^* \times D^*$ defined by

$$x \preceq y \quad : \text{iff} \quad y = xz \text{ for some } z \in D^*.$$

An *unlabelled tree* is a structure $\mathfrak{t}$ isomorphic to $\langle T, \preceq \rangle$ where $T \subseteq D^*$ is prefix closed, for some set D. A *tree* is a structure of the form $\langle T, \preceq, \bar{P} \rangle$ where $\langle T, \preceq \rangle$ is an unlabelled tree and the P_i are unary predicates.

A tree is *deterministic* if it is of the form $\langle T, \preceq, (\text{child}_d)_{d \in D}, \bar{P} \rangle$ where D is finite and

$$\text{child}_d := \{\, ud \mid u \in D^* \,\}.$$

The *complete binary tree* is $\mathfrak{t}_2 := \big\langle \{0,1\}^*, \text{child}_0, \text{child}_1, \preceq \big\rangle$.

We will consider several logics. Besides *first-order logic* FO we will use *monadic second-order logic* MSO which extends FO by set variables and set quantifiers, *weak monadic second-order logic* WMSO which extends FO by variables for finite sets and the corresponding quantifiers, and *guarded second-order logic* GSO. The syntax of GSO is the same as that of full second-order logic where we allow variables for relations of arbitrary arity and quantification over such variables. The semantics of such a second-order quantifier is as follows (for a more detailed definition see [GHO02]). We call a tuple $\bar{a}$ *guarded* if there exists a relation R_i and a tuple $\bar{c} \in R_i$ such that every component a_i of $\bar{a}$ appears in $\bar{c}$. A relation is guarded if it only contains guarded tuples. We define a formula of the form $\exists S \varphi(S)$ to be true if there exists a guarded relation S satisfying φ. Similarly, $\forall S \varphi(S)$ holds if every guarded relation S satisfies φ. For instance, given a graph $\mathfrak{g} = \langle V, E \rangle$ we can use guarded quantifiers to quantify over sets of edges.

Definition 2.1. Let L and L' be two logics. A (total) unary operation f on structures is (L, L')-*compatible* if, for every sentence $\varphi \in L'$, we can effectively compute a sentence $\varphi^f \in L$ such that

$$f(\mathfrak{a}) \models \varphi \quad \text{iff} \quad \mathfrak{a} \models \varphi^f\,, \quad \text{for every structure } \mathfrak{a}\,.$$

We call f (L, L')-*bicompatible* if, furthermore, for every sentence $\varphi \in L$, we can effectively compute a sentence $\varphi' \in L'$ such that

$$\mathfrak{a} \models \varphi \quad \text{iff} \quad f(\mathfrak{a}) \models \varphi'\,, \quad \text{for every structure } \mathfrak{a}\,.$$

For the case that $L = L'$ we simply speak of L-*compatible* and L-*bicompatible* operations.

The interest in compatible operations is mainly based on the fact that they preserve the decidability of theories.

Lemma 2.2. Let f be a (L, L')-compatible operation. If the L-theory of $\mathfrak{a}$ is decidable then so is the L'-theory of $f(\mathfrak{a})$.

Another natural property of this definition is the ability to compose compatible operations.

Lemma 2.3. If f is an (L, L')-compatible operation and g an (L', L'')-compatible one then $g \circ f$ is (L, L'')-compatible. If f and g are bicompatible then so is $g \circ f$.

3 Operations

In this section we survey various operations on structures and their effect on logical theories (see also [Mak04, Tho97a, Gur85]). We attempt to provide a generic and self-contained panorama. We do not intend to present all results in their strongest and most precise form. For instance, many compatibility statements can be strengthened to compatibility for (i) the bounded quantifier fragments of the corresponding logics; (ii) their extensions by cardinality and counting quantifiers; or (iii) operations depending on parameters. The statements we present could also be refined by studying their complexity in terms of the size of formulae. This goes beyond the scope of this survey.

3.1 Generic operations

We start with *interpretations,* which are among the most versatile operations we will introduce. In fact, all other operations we present are quite limited on their own. Only when combined with an interpretation they reveal their full strength.

Definition 3.1. Let L be a logic and Σ and Γ signatures. An *L-interpretation* from Σ to Γ is a list

$$\mathcal{I} = \big\langle \delta(x), (\varphi_R(\bar{x}))_{R \in \Gamma} \big\rangle$$

of L-formulae over the signature Σ where δ has 1 free (first-order) variable and the number of free variables of φ_R coincides with the arity of R.

Such an interpretation induces an operation mapping a Σ-structure $\mathfrak{a}$ to the Γ-structure

$$\mathcal{I}(\mathfrak{a}) := \langle D, R_0, \dots, R_{r-1} \rangle$$

where

$$D := \big\{ a \in A \bigm| \mathfrak{a} \models \delta(a) \big\} \quad \text{and} \quad R_i := \big\{ \bar{a} \in A^n \bigm| \mathfrak{a} \models \varphi_{R_i}(\bar{a}) \big\}.$$

The *coordinate map* of $\mathcal{I}$ is the function mapping those elements of A that encode an element of $\mathcal{I}(\mathfrak{a})$ to that element. It is also denoted by $\mathcal{I}$.

An L-interpretation with $\delta(x) = \texttt{true}$ is called an *L-expansion.* An *L-marking* is an L-expansion that only adds unary predicates without changing the existing relations of a structure.

Proposition 3.2. Let $\mathcal{I}$ be an L-interpretation where L is one of FO, WMSO, or MSO. For every L-formula $\varphi(\bar{x})$, there exists an L-formula $\varphi^{\mathcal{I}}(\bar{x})$ such that

$$\mathcal{I}(\mathfrak{a}) \models \varphi(\mathcal{I}(\bar{a})) \quad \text{iff} \quad \mathfrak{a} \models \varphi^{\mathcal{I}}(\bar{a}),$$

for all structures $\mathfrak{a}$ and all elements $a_i \in A$ with $\mathfrak{a} \models \delta(a_i)$.

The formula $\varphi^{\mathcal{I}}$ is easily constructed from φ by performing the following operations: (i) replacing every atom $R\bar{x}$ by its definition φ_R; (ii) relativising all first-order quantifiers to elements satisfying δ, and all set quantifiers to sets of such elements.

Corollary 3.3. *L-interpretations are L-compatible if L is one of* FO, WMSO, *or* MSO.

A nice property of interpretations is that they are closed under composition.

Proposition 3.4. Let L be one of FO, WMSO, or MSO. For all L-interpretations $\mathcal{I}$ and $\mathcal{J}$, there exists an L-interpretation $\mathcal{K}$ such that $\mathcal{K} = \mathcal{I} \circ \mathcal{J}$.

The second generic operation we are considering is the quotient operation.

Definition 3.5. Let $\mathfrak{a} = \langle A, \bar{R} \rangle$ be a structure and $\sim$ a binary relation. If $\sim$ is a congruence relation of $\mathfrak{a}$ then we can form the *quotient* of $\mathfrak{a}$ by $\sim$ which is the structure

$$\mathfrak{a}/{\sim} := \langle A/{\sim}, \bar{S} \rangle$$

where, if we denote the $\sim$-class of a by $[a]$, we have

$$\begin{aligned} A/{\sim} &:= \{\, [a] \mid a \in A \,\}\,, \\ S_i &:= \{\, \langle [a_0], \dots, [a_{n-1}] \rangle \mid \langle a_0, \dots, a_{n-1} \rangle \in R_i \,\}\,. \end{aligned}$$

By convention, if $\sim$ is not a congruence, we set $\mathfrak{a}/{\sim}$ to be $\mathfrak{a}$.

We will only consider quotients by relations $\sim$ that are already present in the structure. This is no loss of generality since we can use a suitable interpretation to add any definable equivalence relation. For a relation symbol R and a structure $\mathfrak{a}$ we denote by $R^{\mathfrak{a}}$ the relation of $\mathfrak{a}$ corresponding to R.

Proposition 3.6. Let L be one of FO, WMSO or MSO, and $\sim$ a binary relation symbol. The quotient operation $\mathfrak{a} \mapsto \mathfrak{a}/{\sim^{\mathfrak{a}}}$ is L-compatible.

Remark 3.7. (a) The convention in the case that $\sim$ is not a congruence causes no problems for the logics we are considering since in each of them we can express the fact that a given binary relation is a congruence.

(b) In order to factorise by a definable congruence relation that is not present in the structure we can precede the quotient operation by a suitable interpretation that expands the structure by the congruence.

(c) It is also possible to define quotients with respect to equivalence relations that are no congruences. This case is also subsumed by our definition since, given an equivalence relation $\sim$, we can use an FO-interpretation $\mathcal{I}$ to modify the relations of a structure $\mathfrak{a}$ in such a way that $\sim$ becomes a congruence and the quotient $\mathcal{I}(\mathfrak{a})/{\sim}$ equals $\mathfrak{a}/{\sim}$.

Another property of the quotient operation is that it commutes with interpretations in the sense of the following proposition.

Proposition 3.8. Let L be one of FO, WMSO or MSO. For every L-interpretation $\mathcal{I}$ and each binary relation symbol $\sim$, there exists an L-interpretation $\mathcal{J}$ such that

$$\mathcal{I}(\mathfrak{a}/{\sim^{\mathfrak{a}}}) = \mathcal{J}(\mathfrak{a})/{\sim^{\mathcal{J}(\mathfrak{a})}}\,, \quad \text{for every structure } \mathfrak{a}\,.$$

In combination with Proposition 3.4, it follows that every sequence of L-interpretations and quotients can equivalently be written as a single L-interpretation followed by a quotient. This is the reason why one often defines a more general notion of an interpretation that combines the simple interpretations above with a quotient operation by a definable congruence. It follows that these generalised interpretations are also closed under composition.

3.2 Monadic second-order logic

We now turn to operations compatible specifically with monadic second-order logic. The simplest one is the disjoint union. We also present a much more general kind of union called a *generalised sum.* Finally we present Muchnik's iteration construction.

Definition 3.9. The *disjoint union* of two structures $\mathfrak{a} = \langle A, \bar{R} \rangle$ and $\mathfrak{b} = \langle B, \bar{S} \rangle$ is the structure

$$\mathfrak{a} \uplus \mathfrak{b} := \langle A \mathbin{\dot\cup} B, \bar{T} \rangle \quad \text{where} \quad T_i := R_i \mathbin{\dot\cup} S_i \,.$$

The theory of the sum can be reduced to the theory of the two arguments using the following proposition.

Proposition 3.10. Let L be one of FO, MSO, WMSO or GSO. For every L-formula φ there exist L-formulae $\psi_0, \ldots, \psi_n$ and $\vartheta_0, \ldots, \vartheta_n$ such that

$$\mathfrak{a} \uplus \mathfrak{b} \models \varphi \quad \text{iff} \quad \text{there is some } i \leq n \text{ such that } \mathfrak{a} \models \varphi_i \text{ and } \mathfrak{b} \models \vartheta_i \,.$$

Unions behave well with respect to MSO, but the same does not hold for products. A notable exception are products with a fixed finite structure. In the following definition we introduce the simpler product with a finite set, which, up to FO-interpretations, is equivalent to using a finite structure.

Definition 3.11. Let $\mathfrak{a} = \langle A, \bar{R} \rangle$ be a structure and $k < \omega$ a number. The *product* of $\mathfrak{a}$ with k is the structure

$$k \times \mathfrak{a} := \left\langle [k] \times A, \bar{R}', \bar{P}, I \right\rangle,$$

where

$$\begin{aligned} R'_j &:= \{ (\langle i, a_0 \rangle, \ldots, \langle i, a_{n-1} \rangle) \mid \bar{a} \in R_j \text{ and } i < k \}, \\ P_i &:= \{i\} \times A \,, \\ I &:= \{ (\langle i, a \rangle, \langle j, a \rangle) \mid a \in A,\ i, j < k \} \,. \end{aligned}$$

Proposition 3.12. For every MSO-formula $\varphi(X^0, \ldots, X^{n-1})$ and all $k < \omega$, there exists an MSO-formula $\varphi_k(\bar{X}^0, \ldots, \bar{X}^{n-1})$ such that

$$k \times \mathfrak{a} \models \varphi(P^0, \ldots, P^{n-1}) \quad \text{iff} \quad \mathfrak{a} \models \varphi_k(\bar{Q}^0, \ldots, \bar{Q}^{n-1}) \,,$$

where $Q^l_i := \{ a \in A \mid \langle i, a \rangle \in P^l \}$. The same holds for WMSO.

This result can be proven as a consequence of Theorem 3.16 below.

Corollary 3.13. For $k < \omega$, the product operation $\mathfrak{a} \mapsto k \times \mathfrak{a}$ is MSO-compatible. It is MSO-bicompatible if $k \neq 0$. The same holds for WMSO and GSO.

Finite products are sometimes combined with MSO-interpretations resulting in what is called a *parameterless* MSO-*transduction* [Cou94]. Such a transduction maps a structure $\mathfrak{a}$ to the structure $\mathcal{I}(k \times \mathfrak{a})$, where k is a natural number, and $\mathcal{I}$ is an MSO-interpretation. It follows that parameterless MSO-transductions are MSO-compatible. Furthermore, they are closed under composition since, for every MSO-interpretation $\mathcal{J}$, there exists an MSO-interpretation $\mathcal{K}$ with

$$k \times \mathcal{J}(l \times \mathfrak{a}) \cong \mathcal{K}(kl \times \mathfrak{a})\,.$$

The operation of disjoint union can be generalised to a union of infinitely many structures. Furthermore, we can endow the index set with a structure of its own. This operations also generalises the product with a finite structure.

Definition 3.14. Let $\mathfrak{i} = \langle I, \bar{S} \rangle$ be a structure and $(\mathfrak{a}^{(i)})_{i \in I}$ a sequence of structures $\mathfrak{a}^{(i)} = \langle A^{(i)}, \bar{R}^{(i)} \rangle$ indexed by elements i of $\mathfrak{i}$.

The *generalised sum* of $(\mathfrak{a}^{(i)})_{i \in I}$ is the structure

$$\sum_{i \in \mathfrak{i}} \mathfrak{a}^{(i)} := \big\langle U, \sim, \bar{R}', \bar{S}' \big\rangle$$

with universe $U := \{\, \langle i, a \rangle \mid i \in I,\ a \in A^{(i)} \,\}$ and relations

$$\begin{aligned}
\langle i, a \rangle \sim \langle j, b \rangle \quad &: \text{iff} \quad i = j\,, \\
R'_l &:= \big\{\, (\langle i, a_0 \rangle, \dots, \langle i, a_{n-1} \rangle) \bigm| i \in I \text{ and } \bar{a} \in R^{(i)}_l \,\big\}\,, \\
S'_l &:= \big\{\, (\langle i_0, a_0 \rangle, \dots, \langle i_{n-1}, a_{n-1} \rangle) \bigm| \bar{\imath} \in S_l \,\big\}\,.
\end{aligned}$$

To illustrate the definition let us show how a generalised sum can be used to define the standard ordered sum of linear orderings.

Example 3.15. Let $\mathfrak{i} = \langle I, \sqsubset \rangle$ and $\mathfrak{a}^{(i)} = \langle A^{(i)}, <^{(i)} \rangle$, for $i \in I$, be linear orders. Then

$$\sum_{i \in \mathfrak{i}} \mathfrak{a}^{(i)} = \langle U, \sim, <, \sqsubset \rangle$$

where $U = \{\, \langle i, a \rangle \mid a \in A^{(i)} \,\}$ and we have

$$\begin{aligned}
\langle i, a \rangle < \langle j, b \rangle \quad &\text{iff} \quad i = j \text{ and } a <^{(i)} b\,, \\
\langle i, a \rangle \sqsubset \langle j, b \rangle \quad &\text{iff} \quad i \sqsubset j\,.
\end{aligned}$$

If we introduce the new (definable) relation

$$\langle i, a\rangle \prec \langle j, b\rangle \quad :\text{iff} \quad \langle i, a\rangle \sqsubset \langle j, b\rangle \text{ or } \langle i, a\rangle < \langle j, b\rangle$$

then the structure $\langle U, \prec\rangle$ is isomorphic to the ordered sum of the orders $\mathfrak{a}^{(i)}$.

The generalisation of Proposition 3.10 takes the following form.

Theorem 3.16. For every MSO-sentence φ, we can construct a finite sequence of MSO-formulae $\chi_0, \dots, \chi_{s-1}$ and an MSO-formula ψ such that

$$\sum_{i \in \mathfrak{i}} \mathfrak{a}^{(i)} \models \varphi \quad \text{iff} \quad \big\langle \mathfrak{i}, [\![\chi_0]\!], \dots, [\![\chi_{s-1}]\!]\big\rangle \models \psi\,,$$

where $[\![\chi]\!] := \{\, i \in I \mid \mathfrak{a}^{(i)} \models \chi \,\}$.

Remark 3.17. This theorem is a special case of a result of Shelah [She75] following the ideas developped by Feferman and Vaught [FV59], see [Tho97a, Gur85] for a readable exposition. As mentioned above it implies Proposition 3.10 (for MSO) as well as Proposition 3.12.

We finally survey the iteration operation originally introduced by Muchnik. Given a structure $\mathfrak{a}$ this operation produces a structure consisting of infinitely many copies of $\mathfrak{a}$ arranged in a tree-like fashion.

Definition 3.18. The *iteration* of a structure $\mathfrak{a} = \langle A, \bar{R}\rangle$ is the structure $\mathfrak{a}^* := \langle A^*, \preceq, \mathrm{cl}, \bar{R}^*\rangle$ where $\preceq$ is the prefix ordering and

$$\begin{aligned} \mathrm{cl} &:= \{\, waa \mid w \in A^*,\ a \in A \,\}\,, \\ R_i^* &:= \{\, (wa_0, \dots, wa_r) \mid w \in A^*,\ \bar{a} \in R_i \,\}\,. \end{aligned}$$

Theorem 3.19 (Muchnik)**.** The iteration operation is MSO-bicompatible.[1]

Remark 3.20. The Theorem of Muchnik was announced without proof in [Sem84]. The first published proof, based on automata-theoretic techniques, is due to Walukiewicz [Wal96, Wal02]. An exposition can be found in [BB02] and a generalisation to various other logics is given in [BK05].

Example 3.21. (a) Let $\mathfrak{a} := \langle [2], P_0, P_1\rangle$ be a structure with two elements and unary predicates $P_0 := \{0\}$ and $P_1 := \{1\}$ to distinguish them. Its iteration $\mathfrak{a}^* = \langle [2]^*, \preceq, \mathrm{cl}, P_0^*, P_1^*\rangle$ resembles the complete binary tree $\mathfrak{t}_2$. Applying a simple (quantifier free) FO-interpretation $\mathcal{I}$ we obtain $\mathfrak{t}_2 = \mathcal{I}(\mathfrak{a}^*)$.

[1] In the printed version of the paper the same statement is made for WMSO. At present, the question whether the iteration operation is WMSO-(bi)compatible is open. We thank Dietrich Kuske for pointing us to this mistake.

(b) Let $\mathfrak{g}$ be a graph. The *unravelling* of $\mathfrak{g}$ is the graph $\mathcal{U}(\mathfrak{g}) := \langle U, F \rangle$ where U is the set of all paths through $\mathfrak{g}$ and F consists of all pairs $\langle u, v \rangle$ such that the path v is obtained from u by appending a single edge of $\mathfrak{g}$.

The unravelling of $\mathfrak{g}$ can be obtained from $\mathfrak{g}$ via an iteration followed by an interpretation. Note that $\mathfrak{g}^*$ consists of all sequences of vertices of $\mathfrak{g}$. All that is needed to get $\mathcal{U}(\mathfrak{g})$ is to define the subset of those sequences that are paths through $\mathfrak{g}$. This can be done by the formula

$$\delta(w) := \forall u \forall v \big[\mathrm{suc}(u, v) \wedge v \preceq w \rightarrow \exists u' (\mathrm{suc}(u, u') \wedge \mathrm{cl}(u') \wedge E^* u' v)\big] .$$

In view of the examples above we directly obtain the following corollaries.

Corollary 3.22 (Rabin [Rab69]). The MSO-theory of the infinite binary tree $\mathfrak{t}_2$ is decidable.

Corollary 3.23 (Courcelle, Walukiewicz [CW98]). The unravelling operation $\mathcal{U}$ is MSO-compatible and WMSO-compatible.

Finally, let us mention that iterations commute with interpretations in the following sense.

Lemma 3.24 ([Blu03]). For every MSO-interpretation $\mathcal{I}$, there exists an MSO-interpretation $\mathcal{J}$ such that

$$\mathcal{I}(\mathfrak{a})^* = \mathcal{J}(\mathfrak{a}^*), \quad \text{for all structures } \mathfrak{a} .$$

3.3 First-order logic

In this section we concentrate on first-order logic. We start by introducing the power-set operation which relates MSO-theories to FO-theories. This operation provides a systematic way to relate results about FO-compatibility to those about MSO-compatibility above.

Definition 3.25. Let $\mathfrak{a} = \langle A, \bar{R} \rangle$ be a structure. The *power set* of $\mathfrak{a}$ is the structure

$$\mathcal{P}(\mathfrak{a}) := \langle \mathcal{P}(A), \bar{R}', \subseteq \rangle ,$$

where $R'_i := \{ \langle \{a_0\}, \dots, \{a_{n-1}\} \rangle \mid \bar{a} \in R_i \}$.

The *weak power set* $\mathcal{P}_{\mathrm{w}}(\mathfrak{a})$ of $\mathfrak{a}$ is the substructure of $\mathcal{P}(\mathfrak{a})$ induced by the set of all *finite* subsets of A.

Since elements of $\mathcal{P}(\mathfrak{a})$ are sets of elements of $\mathfrak{a}$, FO-formulae over $\mathcal{P}(\mathfrak{a})$ directly correspond to MSO-formulae over $\mathfrak{a}$ (and similarly for WMSO).

Proposition 3.26. (a) For every FO-formula $\varphi(\bar{x})$, we can construct an MSO-formula $\varphi'(\bar{X})$ such that

$$\mathcal{P}(\mathfrak{a}) \models \varphi(\bar{P}) \quad \text{iff} \quad \mathfrak{a} \models \varphi'(\bar{P}) ,$$

for every structure $\mathfrak{a}$ and all subsets $P_i \subseteq A$.

(b) For every MSO-formula $\varphi(\bar{X})$, we can construct an FO-formula $\varphi'(\bar{x})$ such that

$$\mathfrak{a} \models \varphi(\bar{P}) \quad \text{iff} \quad \mathcal{P}(\mathfrak{a}) \models \varphi'(\bar{P}),$$

for every structure $\mathfrak{a}$ and all subsets $P_i \subseteq A$.

(c) Analogous statements hold for WMSO-formulae and the weak power-set operation $\mathcal{P}_{\mathrm{w}}$.

Corollary 3.27. The power-set operation $\mathcal{P}$ is (MSO, FO)-bicompatible and the weak power-set operation $\mathcal{P}_{\mathrm{w}}$ is (WMSO, FO)-bicompatible.

Lemma 3.28. For every MSO-interpretation $\mathcal{I}$, there exists an FO-interpretation $\mathcal{J}$ such that

$$\mathcal{P} \circ \mathcal{I} = \mathcal{J} \circ \mathcal{P}.$$

A similar statement holds with WMSO instead of MSO and $\mathcal{P}_{\mathrm{w}}$ instead of $\mathcal{P}$.

Remark 3.29. In [Col04, CL07b] (*finite*) *set interpretations* are introduced which are halfway between first-order and monadic second-order interpretations. A (finite) set interpretation is of the form

$$\mathcal{I} = \langle \delta(X), (\varphi_R(\bar{X}))_{R \in \Gamma} \rangle$$

where δ, φ_R are (weak) monadic second-order formulae with *set variables* as free variables. Correspondingly the elements of the structure $\mathcal{I}(\mathfrak{a})$ are encoded by (finite) subsets of the original structure. With the operations of the present article we can express such a set interpretation as, respectively,

$$\mathcal{J} \circ \mathcal{P} \quad \text{or} \quad \mathcal{J} \circ \mathcal{P}_{\mathrm{w}}$$

where $\mathcal{J}$ is an FO-interpretation. From Corollary 3.27 it follows that

- set interpretations are (MSO, FO)-compatible and
- finite set interpretations are (WMSO, FO)-compatible.

From Lemma 3.28 and Proposition 3.4 it follows that, if $\mathcal{I}$ is an FO-interpretation, $\mathcal{J}$ a set interpretation, and $\mathcal{K}$ an MSO-interpretation then their composition $\mathcal{I} \circ \mathcal{J} \circ \mathcal{K}$ is also a set interpretation. The same holds for finite set interpretations provided $\mathcal{K}$ is a WMSO-interpretation.

We have mentioned that products are not compatible with monadic second-order logic. But they are compatible with first-order logic. In fact, historically they were among the first operations shown to be compatible with some logic.

Definition 3.30. The (*direct, or Cartesian*) *product* of two structures $\mathfrak{a} = \langle A, \bar{R}\rangle$ and $\mathfrak{b} = \langle B, \bar{S}\rangle$ is the structure

$$\mathfrak{a} \times \mathfrak{b} := \langle A \times B, \bar{T}\rangle\,,$$

where $T_i := \big\{\, (\langle a_0, b_0\rangle, \dots, \langle a_{n-1}, b_{n-1}\rangle) \bigm| \bar{a} \in R_i \text{ and } \bar{b} \in S_i \,\big\}$.

Proposition 3.31. For every FO-formula φ, we can construct FO-formulae $\psi_0, \dots, \psi_n$ and $\vartheta_0, \dots, \vartheta_n$ such that

$$\mathfrak{a} \times \mathfrak{b} \models \varphi \quad \text{iff} \quad \text{there is some } i \leq n \text{ such that } \mathfrak{a} \models \psi_i \text{ and } \mathfrak{b} \models \vartheta_i\,.$$

Product and disjoint union are related via the power-set construction.

Proposition 3.32. There exist FO-interpretations $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ such that

$$\mathcal{P}(\mathfrak{a} \uplus \mathfrak{b}) \cong \mathcal{I}(\mathcal{J}(\mathcal{P}(\mathfrak{a})) \times \mathcal{K}(\mathcal{P}(\mathfrak{b})))\,, \quad \text{for all structures } \mathfrak{a} \text{ and } \mathfrak{b}\,.$$

A similar statement holds with $\mathcal{P}_\mathrm{w}$ instead of $\mathcal{P}$.

Remark 3.33. (a) The interpretations $\mathcal{J}$ and $\mathcal{K}$ are only needed to avoid problems with empty relations. If a relation is empty in one of the factors then the corresponding relation of the product is also empty and cannot be reconstructed. The quantifier-free interpretations are used to create dummy relations to avoid this phenomenon.

(b) Using this result together with the (MSO, FO)-bicompatibility of $\mathcal{P}$ we can deduce the MSO variant of Proposition 3.10 from Proposition 3.31. A similar argument yields the WMSO version.

Similar to finite products that are MSO-compatible we can define a finite exponentiation which is FO-compatible.

Definition 3.34. Let $\mathfrak{a} = \langle A, \bar{R}\rangle$ be a structure and $k < \omega$ a number. The *exponent* of $\mathfrak{a}$ to the k is the structure

$$\mathfrak{a}^k := \langle A^k, \bar{R}', \bar{E}\rangle$$

with relations

$$\begin{aligned} R'_{li} &:= \{\, (\bar{a}^0, \dots, \bar{a}^{n-1}) \mid (a_i^0, \dots, a_i^{n-1}) \in R_l \,\}\,, \\ E_{ij} &:= \{\, (\bar{a}, \bar{b}) \mid a_i = b_j \,\}\,. \end{aligned}$$

The good behaviour of the finite exponent operation is illustrated by the next proposition.

Proposition 3.35. For each $k < \omega$ and every FO-formula $\varphi(x^0, \dots, x^{n-1})$, there exists an FO-formula $\varphi_k(\bar{x}^0, \dots, \bar{x}^{n-1})$ such that

$$\mathfrak{a}^k \models \varphi(\bar{a}^0, \dots, \bar{a}^{n-1}) \quad \text{iff} \quad \mathfrak{a} \models \varphi_k(\bar{a}^0, \dots, \bar{a}^{n-1}),$$

for every structure $\mathfrak{a}$ and all $\bar{a}^i \in A^k$.

Corollary 3.36. Let $k < \omega$. The exponent operation $\mathfrak{a} \mapsto \mathfrak{a}^k$ is FO-compatible. It is FO-bicompatible for $k \geq 1$.

The relation between finite exponentiation and finite products is given in the next proposition. (This allows us to deduce Proposition 3.12 from Proposition 3.35).

Proposition 3.37. For every $k < \omega$, there exists an FO-interpretation $\mathcal{I}$ such that

$$\mathcal{P}(k \times \mathfrak{a}) \cong \mathcal{I}(\mathcal{P}(\mathfrak{a})^k), \quad \text{for every structure } \mathfrak{a}.$$

The same holds for the weak power-set operation.

In the same way as the combination of MSO-interpretations and finite products leads to the notion of a parameterless MSO-transduction, one can perform a finite exponentiation before an FO-interpretation. The resulting operation is called a *k-dimensional* FO-*interpretation.* The composition of a k-dimensional FO-interpretation with an l-dimensional one yields a kl-dimensional FO-interpretation. In the same spirit as above, multi-dimensional interpretations are correlated to parameterless MSO-transductions via the power-set operation.

As for unions we can generalise products to infinitely many factors. In the original definition of a generalised product by Feferman and Vaught [FV59] FO-formulae are used to determine the relations in the product structure. We will adopt a simpler yet richer definition where the product structure is completely determined by the index structure and the factors.

Definition 3.38. Let $\mathfrak{a}^{(i)} = \langle A^{(i)}, \bar{R}^{(i)} \rangle$, $i \in I$, be structures, and let

$$\mathfrak{i} = \langle \mathcal{P}(I), \subseteq, \bar{S} \rangle$$

be the expansion of the power-set algebra $\mathcal{P}(I)$ by arbitrary relations $\bar{S}$. We define the *generalised product* of the $\mathfrak{a}^{(i)}$ over $\mathfrak{i}$ to be the structure

$$\prod_{i \in \mathfrak{i}} \mathfrak{a}^{(i)} := \langle U, \subseteq, \bar{S}, \bar{R}', E_= \rangle$$

with universe

$$U := \mathcal{P}(I) \cup \prod_{i \in I} A^{(i)}$$

where the relations $\subseteq$ and $\bar{S}$ are those of $\mathfrak{i}$ and we have

$$R'_k := \{ (X, \bar{a}) \mid X = [\![R\bar{a}]\!] \},$$
$$E_= := \{ (X, a, b) \mid X = [\![a = b]\!] \},$$

and $[\![\chi(\bar{a})]\!] := \{ i \in I \mid \mathfrak{a}^{(i)} \models \chi(\bar{a}^{(i)}) \}$.

Before stating that the generalised products are compatible with first-order logic let us give two examples.

Example 3.39. Let $\mathfrak{g}_0 = \langle V_0, E_0 \rangle$ and $\mathfrak{g}_1 = \langle V_1, E_1 \rangle$ be two directed graphs. There are two standard ways to form their product: we can take the *direct* or *synchronous* product with edge relation $E_{\mathrm{s}} := E_0 \times E_1$, and we can take the *asynchronous* product with edge relation $E_{\mathrm{a}} := (E_0 \times \mathrm{id}) \cup (\mathrm{id} \times E_1)$. Both kinds of products can be obtained from the generalised product via a first-order interpretation.

For the direct product, we define the edge relation by the formula

$$\varphi_{E_{\mathrm{s}}}(x, y) := \exists z[\mathrm{All}(z) \wedge Ezxy]$$

where the formula

$$\mathrm{All}(x) := x \subseteq x \wedge \forall y(x \subseteq y \rightarrow x = y)$$

states that $x = I$ is the maximal element of $\mathcal{P}(I)$. (Note that the condition $x \subseteq x$ is needed to ensure that $x \in \mathcal{P}(I)$.)

Similarly, we define the edge relation of the asynchronous product by

$$\varphi_{E_{\mathrm{a}}}(x, y) := \exists u \exists v \big[E_= vxy \wedge Euxy \wedge \mathrm{Sing}(u) \wedge \forall z(z \subseteq z \rightarrow (u \nsubseteq z \leftrightarrow z \subseteq v))\big]$$

where the formula

$$\mathrm{Sing}(z) := z \subseteq z \wedge \forall u \forall v[v \subseteq u \subseteq z \rightarrow (v = u \vee u = z)]$$

states that z is a singleton set in $\mathcal{P}(I)$.

Theorem 3.40 (Feferman-Vaught [FV59])**.** For every FO-sentence φ, there exist an FO-sentence φ' and a finite sequence of FO-sentences $\chi_0, \dots, \chi_m$ such that

$$\prod_{i \in \mathfrak{i}} \mathfrak{a}^{(i)} \models \varphi \quad \text{iff} \quad \langle \mathfrak{i}, [\![\chi_0]\!], \dots, [\![\chi_m]\!] \rangle \models \varphi',$$

where $[\![\chi]\!] := \{ i \in I \mid \mathfrak{a}^{(i)} \models \chi \}$.

Remark 3.41. (a) If the structure $\mathfrak{i}$ is of the form $\mathcal{P}(\mathfrak{j})$, for some index structure $\mathfrak{j}$, then instead of a FO-formula φ' over $\mathfrak{i}$ we can also construct an MSO-formula over $\mathfrak{j}$, by Proposition 3.26. Hence, in this case we can reduce the FO-theory of the product $\prod_i \mathfrak{a}^i$ to the MSO-theory of the index structure $\mathfrak{j}$.

(b) Note that Theorem 3.16 follows from Theorem 3.40 and Proposition 3.26 since there exist FO-interpretations $\mathcal{I}, \mathcal{J}$ such that

$$\mathcal{P}\Big(\sum_{i\in\mathfrak{i}} \mathfrak{a}_i\Big) = \mathcal{I}\Big(\prod_{i\in\mathcal{P}(\mathfrak{i})} \mathcal{J}(\mathcal{P}(\mathfrak{a}_i))\Big)\,.$$

(c) As an application of the generalised product we give an alternative proof to a result of Kuske and Lohrey [KL06] which states that, if we modify the iteration operation by omitting the clone relation cl then the resulting operation is (FO, Chain)-compatible. Here, Chain denotes the restriction of MSO where set variables only range over *chains,* i.e., sets that are totally ordered with respect to the prefix order $\preceq$. Let us denote by $\mathfrak{a}^\sharp$ the iteration of $\mathfrak{a}$ without cl and let $\mathcal{P}_{\mathrm{ch}}(\mathfrak{a})$ be the substructure of $\mathcal{P}(\mathfrak{a})$ induced by all chains of $\mathfrak{a}$ (we assume that $\mathfrak{a}$ contains a partial order $\preceq$). A closer inspection reveals that, up to isomorphism, the structure $\mathcal{P}_{\mathrm{ch}}(\mathfrak{a}^\sharp)$ can be obtained by a (2-dimensional) FO-interpretation from the generalised product of several copies of $\mathfrak{a}$ indexed by the structure $\mathcal{P}\langle\omega,<\rangle$. By Theorem 3.40 and the decidability of the MSO-theory of $\langle\omega,<\rangle$ [Büc62], it follows that the operation $\mathfrak{a}\mapsto\mathfrak{a}^\sharp$ is (FO, Chain)-compatible.

3.4 Guarded second-order logic

We conclude this section by considering an operation that connects guarded second-order logic with monadic second-order logic.

Definition 3.42. The *incidence structure* of a structure $\mathfrak{a} = \langle A, R_0, \dots, R_r\rangle$ is

$$\mathrm{In}(\mathfrak{a}) := \langle A \cup G, \bar{R}', I_0, \dots, I_{n-1}\rangle$$

where $G := R_0 \cup \cdots \cup R_r$ is the set of all tuples appearing in a relation of $\mathfrak{a}$, we have *unary* predicates

$$R'_i := \{\, \bar{a} \in G \mid \bar{a} \in R_i \,\}\,,$$

and binary incidence relations $I_i \subseteq A \times G$ with

$$I_i := \{\, (a_i, \bar{a}) \in A \times G \mid \bar{a} \in G \,\}\,.$$

Example 3.43. The incidence structure of a graph $\mathfrak{g} = \langle V, E\rangle$ is the structure

$$\mathrm{In}(\mathfrak{g}) = \langle V \cup E, E', I_0, I_1\rangle$$

where the universe consists of all vertices and edges, the unary predicate E' identifies the edges, and the incidence relations I_0 and I_1 map each edge to its first and second vertex, respectively.

The GSO-theory of a structure is equivalent to the MSO-theory of its incidence structure.

Proposition 3.44. The operation In is (GSO, MSO)-bicompatible.

Remark 3.45. For the proof, note that we can encode every guarded n-tuple $\bar{a}$ by a triple $\langle R, \bar{c}, \sigma \rangle$ consisting of an m-ary relation R, a tuple $\bar{c} \in R$, and the function $\sigma : [n] \to [m]$ such that $a_i = c_{\sigma(i)}$. Consequently, we can encode a guarded relation $S \subseteq A^n$ by a (finite) family of subsets $P_{R,\sigma} \subseteq G$ where

$$P_{R,\sigma} := \{\, \bar{c} \in G \mid \langle R, \bar{c}, \sigma \rangle \text{ encodes an element of } S \,\}.$$

4 Structural properties

So far, we have presented a number of purely logical properties of operations. In this section, we survey other equivalences which hold under some additional hypothesis on the structures in question. First we study properties specific to trees. Then we present results for uniformly sparse structures. Finally we consider structures interpretable in the weak power set of a tree.

4.1 Tree-interpretable structures

When studying logical theories of trees various tools become available that fail for arbitrary structures. The most prominent example are *automata-theoretic methods.* For instance, one can translate every MSO-formula into an equivalent tree automaton (see [Büc60, TW68, Rab69]). Closer to the topic of the present paper are *composition arguments* which are based on Theorem 3.16 and its variants. Those techniques provide the necessary arguments for the tree-specific statements of the present section.

Definition 4.1. A structure is *tree-interpretable* if it is isomorphic to $\mathcal{I}(t)$ for some MSO-interpretation $\mathcal{I}$ and tree $\mathfrak{t}$.

The notion of tree-interpretability is linked to two complexity measures: the *clique width* [Cou04] (for graphs) and the *partition width* [Blu03, Blu06] (for arbitrary structures). It turns out that a graph/structure is tree-interpretable if and only if its clique width/partition width is finite.

In the definition of tree-interpretable structures, we can require the tree $\mathfrak{t}$ to be deterministic without any effect. We can also replace MSO by WMSO without changing the definition. Our first result implies that the definition still remains equivalent if we use FO instead of MSO.

Theorem 4.2 ([Col07]). For every MSO-interpretation $\mathcal{I}$, there exists an FO-interpretation $\mathcal{J}$ and an MSO-marking $\mathcal{M}$ such that

$$\mathcal{I}(\mathfrak{t}) = (\mathcal{J} \circ \mathcal{M})(\mathfrak{t}), \quad \text{for every tree } \mathfrak{t}.$$

The same holds when MSO is replaced by WMSO.

Indeed, since the class of trees is closed under MSO-markings every tree-interpretable structure can be obtained by an FO-interpretation from a tree. Note that it is mandatory for this result that trees are defined in terms of the prefix order $\preceq$ instead of using just the immediate successor relation.

One motivation for the study of tree-interpretable structures is the fact that this class seems to capture the dividing line between simple and complicated MSO-theories. On the one hand, trees have simple MSO-theories and, therefore, so have all structures that can be interpreted in a tree. Conversely, it is conjectured that the MSO-theory of every structure that is not tree-interpretable is complicated.

Conjecture 4.3 (Seese [See91]). Every structure with a decidable MSO-theory is tree-interpretable.

Currently the best result in this direction was recently obtained by Courcelle and Oum [CO07]. It states that every graph that is not tree-interpretable has an undecidable C_2MSO-theory where C_2MSO is the extension of MSO by predicates for counting modulo 2. Unfortunately their proof appears surprisingly difficult to generalise to arbitrary structures.

One evidence for Seese's conjecture is the fact that the class of tree-interpretable structures is closed under all known MSO-compatible operations.

Proposition 4.4. The class of tree-interpretable structures is closed under (i) disjoint unions, (ii) generalised sums, (iii) finite products, (iv) quotients, (v) MSO-interpretations, and (vi) iterations.

There is no difficulty in proving this proposition. In particular, it is easy to establish that the quotient of a tree-interpretable structure is also tree-interpretable. Indeed it is sufficient to guess a system C of representatives of the equivalence classes. Once we have expanded the tree by this new unary predicate C we can use a simple MSO-interpretation to obtain the quotient. However, if one wants the representatives C to be unique and MSO-definable this becomes impossible. This follows from the following result of Gurevich and Shelah [GS83] (see [CL07a] for a simple proof): There is no MSO-formula $\varphi(x, X)$ such that, for every deterministic tree $\mathfrak{t}$ and all nonempty sets $P \subseteq \mathfrak{t}$, there is a unique element $a \in P$ such that $\mathfrak{t} \models \varphi(a, P)$.

The following theorem circumvents this difficulty. It is more precise than simply claiming the closure under quotients in that it states that we can choose the same deterministic tree. The result is given for FO, but it can also be derived for MSO and WMSO by a direct application of Theorem 4.2.

Theorem 4.5. Let $\mathcal{I}$ be an FO-interpretation and $\sim$ a binary relation symbol. There exists an FO-interpretation $\mathcal{J}$ such that

$$\mathcal{I}(\mathfrak{t})/{\sim^{\mathcal{I}(\mathfrak{t})}} \cong \mathcal{J}(\mathfrak{t})\,, \quad \text{for every deterministic tree } \mathfrak{t}\,.$$

Remark 4.6. For the proof of this result it is sufficient to assign to each $\sim$-class a unique element of the tree in an FO-definable way. First, one maps each class to its infimum (for the prefix order $\preceq$). With this definition several classes might be mapped to the same element. Using a technique similar to the one from [CL07b] it is possible to distribute those elements in a FO-definable way and thereby to transform the original mapping into an injective one.

Another phenomenon is that the iteration and unravelling operations turn out to be equivalent in the context of MSO-interpretations over trees.

Theorem 4.7 ([CW03])**.** There exist MSO-interpretations $\mathcal{I}, \mathcal{J}$ such that

$$\mathfrak{t}^* \cong \mathcal{I}(\mathcal{U}(\mathcal{J}(\mathfrak{t})))\,, \quad \text{for every deterministic tree } \mathfrak{t}\,.$$

The first interpretation $\mathcal{J}$ adds backward edges and loops to every vertex of $\mathfrak{t}$. From the unravelling of this structure we can reconstruct the iteration of $\mathfrak{t}$ by an MSO-interpretation.

4.2 Tree width, uniform sparse structures, and complete bipartite subgraphs

In this section we introduce the *tree width* of a structure, a complexity measure similar to the clique width or partition width, which were related to the notion of tree-interpretability. Intuitively the tree width of a structure measures how much it resembles a tree (see [Bod98] for a survey).

Definition 4.8. Let $\mathfrak{a} = \langle A, \bar{R} \rangle$ be a structure.

(a) A *tree decomposition* of $\mathfrak{a}$ is a family $(U_v)_{v \in T}$ of subsets $U_v \subseteq A$ indexed by an undirected tree T with the following properties:

1. $\bigcup_v U_v = A\,.$
2. For all $\bar{a} \in R_i$ in some relation of $\mathfrak{a}$, there is some $v \in T$ with $\bar{a} \subseteq U_v$.
3. For every element $a \in A$, the set $\{\, v \in T \mid a \in U_v \,\}$ is connected.

(b) The *width* of such a tree decomposition $(U_v)_{v \in T}$ is

$$\sup\{\, |U_v| \mid v \in T \,\}\,.$$

(For aesthetic reasons the width is traditionally defined as supremum of $|U_v| - 1$. We have dropped the -1 since it makes many statements more complicated and omitting it does not influence the results.)

(c) The *tree width* $\operatorname{twd} \mathfrak{a}$ is the minimal width of a tree decomposition of $\mathfrak{a}$.

It turns out that, with respect to tree width, GSO plays a similar role as MSO does with respect to tree-interpretability. The incidence structure allows to go back and forth in this analogy.

Theorem 4.9. A structure $\mathfrak{a}$ has finite tree width iff $\mathrm{In}(\mathfrak{a})$ is tree-interpretable.

The corresponding result for classes of finite structures is due to Courcelle and Engelfriet [CE95]. The same ideas can be used to prove Theorem 4.9. Note that this theorem in particular implies that every structure with finite tree width is tree-interpretable. However the converse does not hold. For instance, the infinite clique is tree-interpretable but its tree width is infinite.

The equivalent of Seese's Conjecture 4.3 for tree width has been proved by Seese.

Theorem 4.10 (Seese [See91])**.** Every structure with a decidable GSO-theory has finite tree width.

The proof is based on the Excluded Grid Theorem of Robertson and Seymour [RS86] and on the fact that the class of all finite grids has an undecidable MSO-theory (see also [Cou95a, Blu03]).

In the remaining of this section, we present two other complexity measures for countable structures: sparsity and the existence of big complete bipartite subgraphs in the Gaifman graph. A structure is uniformly sparse if, in every substructure, the number of guarded tuples is linearly bounded by the size of the substructure.

Definition 4.11. Let $k < \omega$. A structure $\mathfrak{a} = \langle A, R_0, \dots, R_{n-1} \rangle$ is called *uniformly k-sparse* if, for all finite sets $X \subseteq A$ and every $i < n$, we have

$$|R_i|_X| \leq k \cdot |X| .$$

A structure is *uniformly sparse* if it is uniformly k-sparse for some $k < \omega$.

The requirement of uniform sparsity is less restrictive than that of having a finite tree width: every structure of finite tree width is uniformly sparse, but the converse does not hold in general. Consider for instance the infinite grid $\mathbb{Z} \times \mathbb{Z}$ with an edge between (i, k) and (j, l) if $|i - j| + |k - l| = 1$. This graph is uniformly sparse, but has infinite tree width.

The work of Courcelle [Cou03] shows that the property of being uniformly sparse is the correct notion for studying the relationship between GSO and MSO. While, in general, GSO is strictly more expressive than MSO, it collapses to MSO on uniformly sparse structures.

Theorem 4.12. Let $k < \omega$. For every GSO-sentence φ, we can construct an MSO-sentence φ' such that

$$\mathfrak{a} \models \varphi \quad \text{iff} \quad \mathfrak{a} \models \varphi' , \quad \text{for all countable uniformly } k\text{-sparse structures } \mathfrak{a} .$$

The proof of this result relies on the possibility, once k is fixed, to interpret $\mathrm{In}(\mathfrak{a})$ in $n \times \mathfrak{a}$ for a suitably chosen n, provided one has correctly labelled $n \times \mathfrak{a}$ by a certain number of monadic *parameters.* Then Theorem 4.12 follows by Proposition 3.44. This technique is formalised by the following lemma.

Lemma 4.13. For all $k < \omega$, there exist $n < \omega$, an MSO-interpretation $\mathcal{I}$, and an MSO-formula φ such that, for every countable uniformly k-sparse structure $\mathfrak{a}$,

- there exist unary predicates $\bar{P}$ such that $\mathfrak{a} \models \varphi(\bar{P})$ and
- $\mathrm{In}(\mathfrak{a}) = \mathcal{I}(n \times \langle \mathfrak{a}, \bar{P} \rangle)$, for all $\bar{P}$ with $\mathfrak{a} \models \varphi(\bar{P})$.

The last notion we present is based on the Gaifman graph of a structure.

Definition 4.14. Let $\mathfrak{a} = \langle A, R_0, \dots, R_{n-1} \rangle$ be a structure. The *Gaifman graph* of $\mathfrak{a}$ is the undirected graph

$$\mathrm{Gaif}(\mathfrak{a}) := \langle A, E \rangle$$

with edge relation

$$E := \{\, (a,b) \mid a \neq b \text{ and } (a,b) \text{ is guarded} \,\} .$$

The Gaifman graph gives an approximation of the relations in a structure. All the notions of this section can be defined in terms of the Gaifman graph as stated by the following proposition.

Proposition 4.15. A structure has finite tree width iff its Gaifman graph has finite tree width. A structure is uniformly sparse iff its Gaifman graph is uniformly sparse.

A *complete bipartite graph* is an undirected graph $\langle V, E \rangle$ where V is partitioned into two sets $A \cup B$ such that

$$E = (A \times B) \cup (B \times A) .$$

If $|A| = |B| = n$ then we say that the graph is of size n. If a graph has complete bipartite subgraphs of arbitrary size this implies that for those subgraphs the number of edges is quadratic in the number of vertices. As a consequence such a graph cannot be uniformly sparse. Hence, for every uniformly sparse graph, there is a bound on the size of its complete bipartite subgraphs. Over structures this means that for every uniformly sparse structure there exists a bound on the size of the complete bipartite subgraphs of its Gaifman graph. However the converse does not hold in general. It is possible to define non-uniformly sparse graphs which do not possess any complete bipartite subgraphs of size larger than some constant. For instance, the graph with vertices $\mathbb{Z}$ with an edge between m and n iff $|m - n|$ is a power of 2.

The three notions of (i) admitting a bound on the size of complete bipartite subgraphs; (ii) being uniformly sparse; and (iii) having bounded tree width; are related but do not coincide. The following theorem states the equivalence of these three notions over tree-interpretable structures. It was first proved for finite graphs in [Cou95b]. The generalisation to infinite structures proceeds along the same lines (see [Blu03]).

Theorem 4.16. For every structure $\mathfrak{a}$, the following statements are equivalent:

1. $\mathfrak{a}$ has finite tree width.
2. $\mathfrak{a}$ is tree-interpretable and uniformly sparse.
3. $\mathfrak{a}$ is tree-interpretable and the size of the complete bipartite subgraphs of its Gaifman graph is bounded.

4.3 The weak power set of trees

We have seen that the power-set construction allows us to relate MSO and FO, in the same way MSO and GSO are related by the incidence structure construction. Hence, one may wonder whether results similar to Theorem 4.10 for GSO or Conjecture 4.3 for MSO hold in this setting. The answer is negative.

Proposition 4.17 ([CL07b])**.** There are structures of decidable FO-theory which are not of the form $\mathcal{I}(\mathcal{P}_{\mathrm{w}}(\mathfrak{t}))$, for a tree $\mathfrak{t}$ and an FO-interpretation $\mathcal{I}$.

An example of this phenomenon is the random graph (a graph in which every finite graph can be embedded) which has a decidable FO-theory but is not of the above form. This propositon is established as an application of the following theorem which eliminates the weak power-set operation in the equation $(\mathcal{I} \circ \mathcal{P}_{\mathrm{w}})(\mathfrak{t}) = \mathcal{P}_{\mathrm{w}}(\mathfrak{a})$, provided that $\mathfrak{t}$ is a deterministic tree.

Theorem 4.18 ([CL07b]). For every FO-interpretation $\mathcal{I}$, there exists a WMSO-interpretation $\mathcal{J}$ such that

$$(\mathcal{I} \circ \mathcal{P}_{\mathrm{w}})(\mathfrak{t}) \cong \mathcal{P}_{\mathrm{w}}(\mathfrak{a}) \quad \text{implies} \quad \mathcal{J}(\mathfrak{t}) \cong \mathfrak{a}\,,$$

for every deterministic tree $\mathfrak{t}$ and every structure $\mathfrak{a}$.

Note that some kind of converse to this theorem can easily be deduced from Lemma 3.28. Indeed, for every WMSO-interpretation $\mathcal{J}$, there exists an FO-interpretation $\mathcal{I}$ such that

$$\mathcal{I} \circ \mathcal{P}_{\mathrm{w}} = \mathcal{P}_{\mathrm{w}} \circ \mathcal{J}\,.$$

Consequently, $\mathcal{J}(\mathfrak{t}) \cong \mathfrak{a}$ implies $(\mathcal{I} \circ \mathcal{P}_{\mathrm{w}})(\mathfrak{t}) = (\mathcal{P}_{\mathrm{w}} \circ \mathcal{J})(\mathfrak{t}) \cong \mathcal{P}_{\mathrm{w}}(\mathfrak{a})$.

Finally, let us state a variant of Theorem 4.5 for the weak power set of a tree.

Theorem 4.19 ([CL07b]). For every FO-interpretation $\mathcal{I}$ and every binary relation symbol $\sim$, there is an FO-interpretation $\mathcal{J}$ such that:

$$(\mathcal{I} \circ \mathcal{P}_{\mathrm{w}})(\mathfrak{t})/{\sim^{(\mathcal{I}\circ\mathcal{P}_{\mathrm{w}})(\mathfrak{t})}} \cong (\mathcal{J} \circ \mathcal{P}_{\mathrm{w}})(\mathfrak{t})\,, \quad \text{for every deterministic tree } \mathfrak{t}\,.$$

When the power set operation is used instead of the weak power-set, we conjecture that this theorem becomes false, whereas Theorem 4.18 remains true: New phenomena arise when infinite sets are allowed.

5 Classes

Suppose that we are interested in, say, the monadic second-order theory of some structure $\mathfrak{a}$. One way to show the decidability of this theory is to start from a structure $\mathfrak{b}$ for which we already know that its monadic second-order theory is decidable, and then to construct $\mathfrak{a}$ from $\mathfrak{b}$ using MSO-compatible operations. We have seen an example of this approach in Corollary 3.22 where the infinite binary tree $\mathfrak{t}_2$ is constructed from a finite structure using an iteration and an MSO-interpretation.

In this last section we follow this idea and consider not only single structures but classes of structures that can be obtained in the way described above. For example, by applying the iteration operation to a finite structure followed by an MSO-interpretation we can not only construct $\mathfrak{t}_2$ but a whole class of structures with a decidable monadic second-order theory. This class and its generalisations are the subject of the first part of this section. In Section 5.2 we consider classes of structures with a decidable first-order theory that can be obtained with the help of FO-interpretations and the (weak) power-set operation. We conclude our survey in Section 5.3 by presenting HR-equational structures and their GSO-theory.

5.1 Prefix-recognisable structures and the Caucal hierarchy

We have conjectured above that all structures with a decidable MSO-theory are tree-interpretable. In this section we take the opposite direction and define large classes of tree-interpretable structures with a decidable MSO-theory. We start with the class of prefix-recognisable structures. Originally, this class was defined as a class of graphs in [Cau96]. These graphs are defined over a universe consisting of a regular set of finite words and their edge relation is given as a finite union of relations of the form

$$(U \times V)W := \{\, (uw, vw) \mid u \in U,\ v \in V,\ w \in W \,\},$$

for regular languages U, V, W. Such relations are a combination of a recognisable relation $U \times V$ for regular U and V, followed by the identity relation, explaining the term 'prefix-recognisable'.

This definition can been extended to arbitrary structures instead of graphs (see [Blu04, CC03]) but the description of prefix-recognisable relations gets more complicated. Using the approach of compatible operations we obtain an alternative and simpler definition of the same class of structures.

Definition 5.1. A structure $\mathfrak{a}$ is *prefix-recognisable* if and only if $\mathfrak{a} \cong \mathcal{I}(\mathfrak{t}_2)$, for some MSO-interpretation $\mathcal{I}$.

This definition directly implies that each prefix-recognisable structure is tree-interpretable and has a decidable monadic second-order theory because $\mathfrak{t}_2$ has. Further elementary properties are summarised in the following proposition.

Proposition 5.2. The class of prefix-recognisable structures is closed under (i) MSO-interpretations, (ii) parameterless MSO-transductions, (iii) disjoint unions, (iv) finite products, (v) quotients, and (vi) generalised sums of the form $\sum_{i \in \mathfrak{i}} \mathfrak{a}$ in which both $\mathfrak{a}$ and $\mathfrak{i}$ are prefix-recognisable and all summands $\mathfrak{a}$ are isomorphic.

In fact, according to Theorem 4.2[2], we can even replace MSO-interpretations by FO-interpretations.

Theorem 5.3 ([Col07]). A structure $\mathfrak{a}$ is prefix-recognisable if and only if $\mathfrak{a} \cong \mathcal{I}(\mathfrak{t}_2)$, for some FO-interpretation $\mathcal{I}$.

For prefix-recognisable graphs several alternative characterisations have been given, for example they are the configuration graphs of pushdown automata after factoring out the ε-transitions, and also those graphs obtained

[2] In combination with the fact that every regular tree is FO-interpretable in $\mathfrak{t}_2$.

as the least solutions of finite systems of equations whose operations consists of (i) disjoint unions and (ii) positive quantifier-free interpretations (this approach is due to Barthelmann [Bar98], see [Blu01] for an overview).

In the definition of prefix-recognisable structures we have used the infinite binary tree $\mathfrak{t}_2$ as a generator and applied MSO-interpretations to it. In Section 3 we have seen how $\mathfrak{t}_2$ can be obtained from a finite structure with the help of the iteration operation. In fact, we do not get more structures when we allow the application of an MSO-interpretation to the iteration of an arbitrary finite structure.

Proposition 5.4. The prefix-recognisable structures are exactly those of the form $\mathcal{I}(\mathfrak{a}^*)$ for an MSO-interpretation $\mathcal{I}$ and a finite structure $\mathfrak{a}$.

As both operations used in Proposition 5.4 are MSO-compatible there is no reason to stop after just one application of each of them. This idea is used in [Cau02] for graphs using the unravelling operation instead of the iteration and an inverse rational mapping (a weakening of an MSO-interpretation) instead of an MSO-interpretation. According to [CW03] the following definition is equivalent to the original one.

Definition 5.5. The *Caucal hierarchy* $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \dots$ is defined as follows. The first level $\mathcal{C}_0$ consists of all finite structures. Each higher level $\mathcal{C}_{n+1}$ consists of all structures of the form $\mathcal{I}(\mathfrak{a}^*)$ where $\mathcal{I}$ is an MSO-interpretation and $\mathfrak{a} \in \mathcal{C}_n$.

The compatibility of the employed operations directly yields the decidability of the MSO-theory for all structures in this class.

Theorem 5.6. All structures in the Caucal hierarchy have a decidable MSO-theory.

In the same spirit as Theorem 5.3 one can show that MSO-interpretations can be replaced by FO-interpretations. Furthermore, the iteration can also be replaced by the unravelling operation applied to the graphs on each level.

Theorem 5.7 ([Col07])**.** A structure belongs to $\mathcal{C}_{n+1}$ if and only if it is of the form $(\mathcal{I} \circ \mathcal{U})(\mathfrak{g})$ where $\mathcal{I}$ is an FO-interpretation, $\mathcal{U}$ the unravelling operation, and $\mathfrak{g} \in \mathcal{C}_n$ is a graph.

At present, the Caucal hierarchy is the largest known natural class of structures with a decidable MSO-theory (other structures with decidable MSO-theory can be constructed by ad hoc arguments; see, e.g., Proposition 5 of [CW03]). The first level of this hierarchy, i.e., the class of prefix-recognisable structures, is already well investigated. In [CW03] the graphs of level n are shown to be the same as the configuration graphs of higher-order pushdown automata of level n (automata using nested stacks of nesting depth n). Using this equivalence and a result on the languages accepted by higher-order pushdown automata, one obtains the strictness of the Caucal hierarchy. In [CW03] it is also shown that not all structures of decidable MSO-theory are captured: There is a tree with decidable MSO-theory that is not contained in any level of the hierarchy. It remains an open task to gain a better understanding of the structures in higher levels of the hierarchy.

5.2 Automatic structures and extensions

Let us turn to structures with a decidable FO-theory. A prominent class of such structures is the class of automatic (and tree-automatic) structures, a notion originally introduced by Hodgson [Hod83].

A relation $R \subseteq (\Sigma^*)^r$ on words is *automatic* if there is a finite automaton accepting exactly the tuples $(w_0, \dots, w_{r-1}) \in R$, where the automaton reads all the words in parallel with the shorter words padded with a blank symbol (for formal definitions see, e.g., [KN95, Blu99, BG00]). A structure is called *automatic* (or has an automatic presentation) if it is isomorphic to a structure whose universe is a regular set of words and whose relations are automatic in the sense described above.

In the same way we can also use automata on finite (ranked) trees to recognise relations. The superposition of a tuple of trees is defined by aligning their roots and then, for each node aligning the sequence of successors from left to right, filling up missing positions with a blank symbol (again, a formal definition can be found in [Blu99, BG00]). Accordingly, a structure is called *tree-automatic* if it is isomorphic to a structure whose domain consists of a regular set of finite trees and whose relations are recognised by finite automata reading the superpositions of tuples of trees. An alternative definition for tree-automatic structures can be given via least solutions of system of equations [Col04] in the same spirit as [Bar98] for prefix-recognisable structures. In addition to the operations for prefix-recognisable structures one allows the Cartesian product in the equations.

By inductively translating formulae to automata we can use the strong closure properties of finite automata to show that each FO-definable relation over an automatic structure is again automatic. As the emptiness problem for finite automata is decidable this yields a decision procedure for the model-checking of FO-formulae over (tree-)automatic structures.

We are interested in generating structures with a decidable FO-theory using FO-compatible operations. We focus here on the use of FO-interpretations. The first possibility is to start from structures with a decidable FO-theory and then apply FO-interpretations to it. Alternatively we can start from structures with a decidable MSO-theory and then apply the (weak) power-set operation followed by an FO-interpretation.

To obtain the class of automatic structures in this way let us first note that each automatic structure can be represented using a binary alphabet, say $[2] = \{0, 1\}$. A word over this alphabet can either be seen as the binary encoding of a number, or as a set of natural numbers, namely the set of all positions in the word that are labelled by 1.

When encoding [2]-words by natural numbers we need relations that allow us to extract single bits of a number to be able to simulate the behaviour of finite automata in first-order logic. This can be done using the addition operation $+$ and the relation $|_2$ defined as follows (see, e.g., [BHMV94, Blu99]):

$$k \mid_2 m \quad : \text{iff} \quad k \text{ is a power of 2 dividing } m\,.$$

Similarly, if [2]-words are viewed as sets of natural numbers we have to be able to access the elements of the set. This is possible in the weak power-set of the structure $\langle \omega, < \rangle$. By Corollary 3.27, FO over $\mathcal{P}_\mathrm{w}\langle \omega, < \rangle$ corresponds to WMSO over $\langle \omega, < \rangle$, which is known to have the same expressive power as finite automata (see, e.g., [Tho97b]).

These ideas lead to the following characterisations of automatic structures.

Proposition 5.8. Let $\mathfrak{a}$ be a structure. The following statements are equivalent:

1. $\mathfrak{a}$ is automatic.
2. $\mathfrak{a} \cong \mathcal{I}\langle \mathbb{N}, +, |_2 \rangle$, for some FO-interpretation $\mathcal{I}$.
3. $\mathfrak{a} \cong (\mathcal{I} \circ \mathcal{P}_\mathrm{w})\langle \omega, < \rangle$, for some FO-interpretation $\mathcal{I}$.

To obtain tree-automatic structures we first note that it is enough to consider unlabelled finite binary trees. Such a tree can be encoded in the infinite binary tree $\mathfrak{t}_2$ by the set of its nodes. It is not difficult to see that first-order logic over the weak power-set structure of $\mathfrak{t}_2$ has the same expressive power as finite automata over trees.

Proposition 5.9. A structure $\mathfrak{a}$ is tree-automatic if and only if $\mathfrak{a} = (\mathcal{I} \circ \mathcal{P}_\mathrm{w})(\mathfrak{t}_2)$, for some FO-interpretation $\mathcal{I}$.

This approach via compatible operations can easily be generalised by using other generators than $\langle \omega, < \rangle$ and $\mathfrak{t}_2$. In the previous section we have obtained a hierarchy of structures with a decidable MSO-theory. The infinite binary tree $\mathfrak{t}_2$ is on the first level of this hierarchy. Using Proposition 5.9 as a definition for tree-automatic structures, we obtain a natural hierarchy of higher-order tree-automatic structures.

Definition 5.10. A *higher-order tree-automatic structure of level n* is a structure of the form $(\mathcal{I} \circ \mathcal{P}_\mathrm{w})(\mathfrak{t})$ for some tree $\mathfrak{t}$ from $\mathcal{C}_n$, the n-th level of the Caucal hierarchy.

Using Theorem 5.6 and the properties of the operations involved we obtain the following result.

Theorem 5.11. Every higher-order tree-automatic structure has a decidable first-order theory.

Although the Caucal hierarchy is known to be strict this does not directly imply that the hierarchy of higher-order tree-automatic structures is also strict. But by Theorem 4.18 it follows that, if the hierarchy would collapse then all the trees in the Caucal hierarchy could be generated from a single tree $\mathfrak{t}$ in this hierarchy by means of WMSO-interpretations. This would contradict the strictness of the Caucal hierarchy because, according to [CW03][3], each level is closed under WMSO-interpretations.

Theorem 5.12 ([CL07b])**.** The hierarchy of higher-order tree-automatic structures is strict.

As mentioned in the previous section very little is known about structures on the higher levels of the Caucal hierarchy. As higher-order tree-automatic structures are defined by means of the Caucal hierarchy we even know less about these structures. In [CL07b] it is illustrated how to apply Theorem 4.18 to show that structures are not higher-order tree-automatic.

5.3 HR-equational structures

In [Cou89] equations using operations on structures are used to define infinite structures. The operations work on structures that are coloured by a finite set of colours. We introduce constant symbols for each finite structure (over a fixed signature). From these we build new structures using:

- the disjoint union operation $\uplus$;
- unary operations ρ_{ab} recolouring all elements of colour a to colour b;
- unary operations θ_a that merge all elements of colour a into a single element.

For example, the equation

$$x = \rho_{20}\big(\theta_2\big((\overset{1}{\bullet}\!\rightarrow\!\overset{2}{\bullet}) \uplus \rho_{12}(x)\big)\big)$$

has as least solution the graph

$$\overset{1}{\bullet}\!\rightarrow\!\overset{0}{\bullet}\!\rightarrow\!\overset{0}{\bullet}\!\rightarrow\!\overset{0}{\bullet}\!\rightarrow \cdots$$

[3] In [CW03] the closure of each level under MSO-interpretations is shown. But in the same paper it is shown that each level can be generated by MSO-interpretations from a deterministic tree of this level, and on deterministic trees the finiteness of a set can be expressed in MSO. Hence the levels are also closed under WMSO-interpretations.

The class of structures obtained as solutions of finite systems of equations over these operations has various names in the literature (equational, regular, hyperedge replacement). We use here the term HR-*equational.*

We obtain a connection between HR-equational structures and trees by unravelling the system of equations defining a given structure $\mathfrak{a}$ into an infinite tree $\mathfrak{t}$. The inner nodes of the tree are labelled with the operations and the leaves with the finite structures that are used as building blocks for the resulting infinite structure. As an unravelling of a finite system of equations the tree $\mathfrak{t}$ is regular and it contains all the information on how to build the structure $\mathfrak{a}$.

It should not be surprising that it is possible to construct the structure $\mathfrak{a}$ from $\mathfrak{t}$ via a parameterless MSO-transduction. But we can do even better because all the information on the relations of $\mathfrak{a}$ is contained in the leaves of the defining tree. This allows us to construct not only $\mathfrak{a}$ but also $\mathrm{In}(\mathfrak{a})$ by a parameterless MSO-transduction. It turns out that this property characterises HR-equational structures. As for prefix-recognisable structures we therefore choose this property as the definition.

Definition 5.13. A structure $\mathfrak{a}$ is HR-*equational* if and only if $\mathrm{In}(\mathfrak{a})$ is prefix-recognisable.

By Proposition 3.44 we can reduce the GSO-theory of an HR-equational structure to the MSO-theory of a prefix-recognisable one.

Proposition 5.14. Every HR-equational structure has a decidable GSO-theory.

Courcelle [Cou90] has proved that the isomorphism problem for HR-equational structures is decidable. We can generalise this result as follows. In [Blu04] it is shown that prefix-recognisable structures can be axiomatised in GSO, i.e., for each prefix recognisable structure $\mathfrak{a}$, one can construct a GSO-sentence $\psi_{\mathfrak{a}}$ such that

$$\mathfrak{b} \models \psi_{\mathfrak{a}} \quad \text{iff} \quad \mathfrak{b} \cong \mathfrak{a}\,, \quad \text{for every structure } \mathfrak{b}\,.$$

If we take $\mathfrak{b}$ from a class of structures for which we can decide whether $\mathfrak{b} \models \psi_{\mathfrak{a}}$ holds then this allows us to solve the isomorphism problem for $\mathfrak{a}$ and $\mathfrak{b}$. To this end let $\mathfrak{b}$ be a uniformly sparse structure from the Caucal hierarchy. (Note that every HR-equational structure is uniformly sparse.) According to Theorem 4.12 we can construct an MSO-sentence $\psi'_{\mathfrak{a}}$ that is equivalent to $\psi_{\mathfrak{a}}$ on $\mathfrak{b}$. And since the MSO-theory of each structure in the Caucal hierarchy is decidable we can now verify if $\mathfrak{b} \models \psi'_{\mathfrak{a}}$, which is the case if, and only if, $\mathfrak{a} \cong \mathfrak{b}$.

Theorem 5.15. Given an HR-equational structure $\mathfrak{a}$ and a uniformly sparse structure $\mathfrak{b}$ from the Caucal hierarchy, we can decide whether $\mathfrak{a} \cong \mathfrak{b}$.

The above description is slightly simplified. The GSO-sentence $\psi_{\mathfrak{a}}$ constructed in [Blu04] uses cardinality quantifiers $\exists^{\kappa}$ meaning "there are at least κ many", for a cardinal κ. To make Theorem 5.15 work in this extended setting, we first note that Theorem 4.12 also works if the logics are extended with cardinality quantifiers. Second, we have to verify that $\mathfrak{b} \models \psi'_{\mathfrak{a}}$ can also be checked if $\psi'_{\mathfrak{a}}$ contains cardinality quantifiers. Because $\mathfrak{b}$ is countable, we only need to consider the quantifier "there are infinitely many". This quantifier can be eliminated since each structure of the Caucal hierarchy can be obtained by an MSO-interpretation from a deterministic tree of the same level and on such trees the property of a set being infinite can be expressed in MSO.

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